Models and Multiplicities

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Abstract: I claim that Wittgenstein's reference to Hertz's 'dynamical models' at 4.04 in the *Tractatus* provides evidence for the view that the *Tractatus* does not explain the sense of propositions by offering an account of the fundamental structure of reality. Just as Hertz's dynamical models capture what all mechanical descriptions of the same system have in common, so Tractarian analysis captures what all propositions that express the same sense have in common, and in neither case is there any need to appeal to an underlying ontology.

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1. Introduction

It is widely accepted that Hertz's *Principles of Mechanics* was one of Wittgenstein's earliest and longest-lasting influences. Wittgenstein cited *Principles* in the *Tractatus* and also considered using a quotation from Hertz's introduction as the motto for the *Philosophical Investigations*.[1] Furthermore, Wittgenstein referred to Hertz both times that he gave a public address at Cambridge,[2] and once wrote: "As I do philosophy, its entire task is to shape expression in such a way that certain worries disappear. ((Hertz.))" (*The Big Typescript*, 310). This last quote points to an aspect of Hertz's influence that is relatively well understood. In *Principles*, Hertz gestures at a subtle ambiguity in Newton's laws of motion, and claims that this ambiguity is responsible for confused questions concerning the "essence" (*Wesen*) of force. Hence an overarching goal of *Principles* is to avoid this ambiguity, so that such confused questions no longer arise.[3] It would be uncontentious to claim that this notion of 'dissolving' a confused question—finding a perspective from which the question no longer seems pressing—had a powerful influence on Wittgenstein's later conception of the ambitions of philosophy. What is less well understood, however, is Hertz's earlier influence on the *Tractatus*.

Interpretations of the *Tractatus* can be divided into two broad camps: *ontologically oriented* and *logically oriented*. These can be characterized by their differing accounts of the central Tractarian claim that propositions with sense are truth-functions of *elementary propositions*; propositions which consist of names of *simple objects*.[4] According to an ontologically oriented interpretation, the *Tractatus* accounts for the sense of propositions by showing that the fundamental structure of language mirrors the fundamental structure of reality: it is because of the correlations between names and simple objects that language can describe the world. According to a logically oriented interpretation, however, the *Tractatus* makes no appeal to a self-standing conception of the fundamental structure of reality, independent of language or thought, in order to account for the sense of propositions. On a logically oriented view, the purpose of re-writing propositions as truth-functions of elementary propositions is simply to make the logical relationships among propositions clear, and thus to avoid certain misunderstandings concerning the logic of our language.[5]

For the purposes of this paper, the key distinction between these interpretations is the following. According to an ontologically oriented interpretation, Tractarian analysis is "bottom-up": the *last* stage of analysis—names occurring in elementary propositions—plays a primary role. In contrast, a logically oriented construal of Tractarian analysis is "top-down": the *first* stage of analysis—ordinary propositions and their manifest logical relationships—is what plays the primary role.[6] To illustrate this distinction, it is particularly helpful to draw a comparison with Russell's logical atomism.[7] At around the time that Wittgenstein was working on the *Tractatus*, Russell was working within an epistemological framework which privileged the notion of *acquaintance*—an unmediated relation between a subject and an object.[8] Sense-data provide particularly vivid examples of objects known by acquaintance—a subject's relation to a patch of red in their visual field (for example) appears especially unmediated—and this gives an intuitive sense of Russellian acquaintance more generally. For Russell, it is impossible for something to be the referent of a logically proper name unless that object is known by acquaintance; hence an important goal of Russellian analysis is to show that ordinary proper names are *not* names in this sense (rather, they are typically disguised definite descriptions). On Russell's view, the analysis of propositions terminates at the level of "atomic" propositions, and the names that appear there will only be logically proper names. Acquaintance with objects imbues these names with meaning, and our ability to understand ordinary propositions stems, ultimately, from our knowledge of objects with which we are acquainted.[9] The *last* stage of analysis thus plays a primary role: the termination of analysis at the level of atomic propositions provides an account for the meaningfulness of ordinary propositions. In this sense, Russellian analysis is bottom-up.

According to an ontologically oriented interpretation, Tractarian analysis is similarly bottom-up. Commentators inclined to an ontologically oriented view find in the *Tractatus* an argument for the necessary existence of simple objects, conceived of as fundamental metaphysical entities which exist independently of language or thought.[10] On this view, it is the simple objects which imbue the names in elementary propositions with meaning, and the sense of ordinary propositions is then accounted for by appealing to the sense of elementary propositions. This is the kind of ontologically oriented understanding of simple objects defended by Peter Hacker:

> The simple objects are, Wittgenstein thought, the final residue of analysis, the indecomposable elements that are the meanings of the unanalysable names that occur in elementary propositions. . . . He *knew*, so he thought, that there must be such things. There must be unanalysable objects if language is to be related to the world. . . For only thus can the need for a firm anchor for language be met.[11]

In contrast with this, a logically oriented interpretation carries with it a top-down conception of analysis: the sense of elementary propositions stems from the sense of ordinary propositions, not the other way around. Thus it is the *first* stage of analysis—propositions with sense and their manifest logical relationships—that plays the primary role. On a logically oriented view, the notion of a simple object only finds its significance given the application and use of language, and the claim that such objects exist does not add anything beyond the claim that propositions have sense; that propositions represent or picture states of affairs. Tractarian simple objects are thus not construed as existing independently of our means of *describing* reality.[12] Although there is more that would need to be said here regarding what a logically oriented understanding of simple objects amounts to, the difference I am interested in concerns whether it is the sense of elementary propositions which accounts for the sense of ordinary propositions, or vice versa. Hence, in the passage from Hacker quoted above, it is really just the final claim ("only thus can the need for a firm anchor for language be met") where a proponent of a logically oriented interpretation would be bound to disagree. A commentator inclined to a logically oriented interpretation would disagree, that is, with the idea that the Tractarian ontology is what somehow grounds the sense of colloquial language.[13]

My central claim in this paper is that Wittgenstein's reference to Hertz's dynamical models at 4.04 is evidence for a logically oriented interpretation.[14] A Hertzian dynamical model captures the *essential content* of a mechanical description—the content that all descriptions of the same system have in common. This has a clear parallel in the *Tractatus*: for Wittgenstein, analysis captures the essential content of a proposition—what all propositions that express the same sense have in common. Furthermore, where *Principles* aims to help us recognize that all mechanical phenomena fall under Hertz's "fundamental law." the *Tractatus* aims to help us recognize that all propositions with sense are instances of Wittgenstein's "general propositional form." I will argue that treating Hertzian analysis as a model of logically oriented analysis provides important insights into how such analysis terminates. Furthermore, I will suggest that this comparison with *Principles* provides insights into what the upshot of the *Tractatus* is supposed to be.

2. Logical Pictures

When Wittgenstein refers to Hertz's dynamical models, he indicates that they are particularly helpful in seeing what it means for a proposition to have the same "multiplicity" as the situation it represents:[15]

> 4.04 There must be just as much that is distinguishable in a proposition as there is in the situation that it represents.
>
> The two must possess the same logical (mathematical) multiplicity. (Compare Hertz's *Mechanics* on dynamical models.)[16]

This remark occurs in Wittgenstein's articulation of the Tractarian picture-theory. There are two sections of the *Tractatus* in which the word 'picture' (*Bild*) occurs in a prominent and sustained way. The first is the series of remarks in the 2.1s and 2.2s, where Wittgenstein articulates the Tractarian conception of picturing in general. The second is the series of remarks in the 4.0s, where Wittgenstein applies this conception of picturing to propositions.

Wittgenstein's description of propositions as pictures is not figurative—he insists that a written proposition proves to be a picture "even in the ordinary sense" (4.011). At 4.012 we are given an indication of how this might work: in a proposition of the form '*aRb*'—which says that *a* stands to *b* in the relation *R*—the propositional sign itself *looks like* two things of a certain kind, '*a* 'and '*b*', related to each other by the fact that '*R* 'stands between them.[17] But it is apparent that a propositional sign like '*aRb* 'is the exception rather than the rule: most declarative propositions don't look anything like the situations they assert to obtain. Although Wittgenstein repeatedly suggests that the operative conception of picturing in the *Tractatus* is, in certain central ways, a familiar one, it is difficult to see how ordinary propositions could possibly be pictures in a familiar sense. Our immediate task is to address this puzzle.

In the 2.1s, we are told that the elements of a picture correspond to objects (2.13), and that what constitutes a picture is that its elements are related to one another in a determinate way (2.14). Following this, the notion of *pictorial form* is introduced:

> 2.15 The fact that the elements of a picture are related to one another in a determinate way represents that things are related to one another in the same way.
>
> Let us call this connection of its elements the structure of the picture, and let us call the possibility of this structure the pictorial form of the picture.

Pictorial form is "the possibility that things are related to one another in the same way as the elements of the picture" (2.151). This is readily applicable to familiar pictures: the spatial relations in a spatial picture, or the relations among colors in a colored picture, mirror the corresponding relations among the depicted objects. Hence a diagram showing a spatial arrangement of furniture could have been drawn with the furniture in different positions, or a painting showing the colors of certain flowers could have been painted with a different palette. There is thus a kind of isomorphism—a shared set of possibilities—between the picture and the situation it represents.[18] According to the *Tractatus*, it is pictorial form which imbues the correlations between the elements of a picture and the associated objects with the significance that they have (2.151–1514); a picture *is* a picture in virtue of the pictorial form that it shares with what it represents (2.16, 2.17).

To make these ideas more concrete, imagine that I want to represent the order in which a truck, car, and taxi are parked outside my house by arranging a cup, book, and pen in a line on my desk.[19] Such a physical model is a particularly direct way of representing a spatial fact—the model can employ exactly the same kind of relations among its elements as the spatial arrangement of objects it depicts because, of course, the model is itself a spatial arrangement of objects. To simplify the example further, imagine that the only fact that I intend to represent is the mere linear ordering of the vehicles—i.e. which one is between the other two—rather than any further facts about their relative locations (which one is furthest to the left, how close together they are, and so on). The pictorial form that allows the items on my desk to represent the order in which the vehicles are parked can be seen in the possibility of arranging them accordingly once I have determined which item is correlated with which vehicle. Hence these two facts—the ordering of the items on my desk and the ordering of the vehicles outside—have the same pictorial form, and the one can represent the other.

So conceived, the same pictorial form can be easily manifested in more abstract representations. All that is required is that the possibilities of the relations among the elements of the representation is the same as the possibilities of the relations among the represented objects. For example, rather than arranging the items on my desk in a line I could instead place a coin on one of them to indicate that it is 'between' the other two. In this case it is the presence of the coin rather than the items' spatial arrangement that represents the ordering of the vehicles. That too would provide a picture of the vehicles by dint of having the same pictorial form; the same possibilities of relations among the elements of the model. (The coin could be placed on any of the items on my desk, thus mirroring the possibility that any of the vehicles is the one parked between the other two.) Beyond this, the same pictorial form could re-emerge in a great variety of abstract representations: in symbols on a piece of paper, in a series of sounds, in a certain pattern of wiggling of my eyebrows, and so on. All that is required is the possibility that the elements of the representation are related to one another in the same way as the depicted objects.[20]

Let us turn to consider how this conception of picturing can be applied to propositions. The claim at 4.01 that a proposition is a picture is justified at 4.02 with the seemingly unrelated observation that "we can understand the sense of a propositional sign without its having been explained to us." Wittgenstein elaborates on this by noting that, although the meaning of an unfamiliar word needs to be explained, the meaning of an unfamiliar *proposition* is readily comprehensible (4.026); a proposition is typically understood by anyone who understands its constituents (4.024). This leads naturally into 4.03, "A proposition must use old expressions to communicate a new sense." But what has all this got to do with the idea that a proposition is a picture?

Wittgenstein's emphasis on the construction of unfamiliar propositions from familiar words becomes intelligible with the central role of pictorial form in view. Roughly, it is the possibilities of the syntactical relations among the elements of a proposition (the words) which can mirror the possibilities of the relations among the elements of the represented situation. Here, then, is how a proposition is a picture. Returning to the example of the vehicles parked outside my house, I can describe the relative positions of the vehicles by saying (for example), 'the car is parked between the truck and the taxi.'[21] The pictorial form that this proposition shares with the situation it depicts can be seen in the possibility of rearranging the sentential elements as required. (Permuting the words 'car' and 'truck' to form a new proposition is like switching around the cup and the book on my desk.) By dint of the fact that it can represent the linear ordering of three individuals, such a proposition then has *the same pictorial form* as the fact that the vehicles in the street (or the items on my desk) are arranged in a particular order.[22] Pictorial form thus provides a central bridge between the Tractarian notion of picturing in general and the application of that notion to propositions. If a proposition is to be a picture of reality, then what it must have in common with reality, in order to be able to depict it in the way it does, is its pictorial form. This is how propositions prove to be pictures even in the ordinary sense.

Note, however, that it is specifically a *logical* picture that is mentioned at 4.03, "A proposition communicates a situation to us, and so it must be *essentially* connected with the situation. And the connection is precisely that it is its logical picture.” The notion of a logical picture is introduced in the concluding remarks of the 2.1s:

> 2.18 What any picture, of whatever form, must have in common with reality, in order to be able to depict it—correctly or incorrectly—in any way at all, is logical form, i.e. the form of reality.
>
> 2.181 A picture whose pictorial form is logical form is called a logical picture.

Logical form is common to all pictures which can depict reality correctly or incorrectly. Given two possible situations, the obtaining or non-obtaining of one may guarantee the obtaining or non-obtaining of the other, or vice versa, or else they may be independent. Thus: the correctness or incorrectness of one picture may imply the correctness or incorrectness of a second picture, or vice versa, or else *they* may be independent.[23] Because every picture that can depict reality correctly or incorrectly sits in its logical relationships with other such pictures, every picture is “*at the same time* a logical one” (2.182).

In the discussion of picturing so far, there has been no need to depart from straightforward and familiar examples, examples in which the depicted objects are just ordinary things like pens, flowers, and taxis. But immediately following the discussion of picturing in the 2.1s, 2.201 declares, “A picture depicts reality by representing a possibility of existence and non-existence of states of affairs.”[24] Tractarian states of affairs are logically independent of one another: the obtaining or non-obtaining of one has no bearing on the obtaining or non-obtaining of any other.[25] This implies that an ordinary fact concerning, say, the relative positions of items on my desk cannot be a state of affairs: if I know that the pen is between the cup and book then I can immediately infer that the cup is not between the book and the pen. Hence commonplace facts clearly fail to meet the independence condition demanded of states of affairs. We are thus faced with the question: why does Wittgenstein claim that an ordinary picture, or an ordinary proposition, is a representation of the obtaining and non-obtaining of logically independent states of affairs?

The answer to this question (or part of an answer) will require us to return, eventually, to the notion of multiplicity and the reference to Hertz's dynamical models at 4.04. But the starting place is the idea that both the ordinary proposition and the corresponding fully analyzed proposition are *projections* of the situation they represent.[26] The term 'projection' is introduced and discussed in the remarks following 3.1 and reoccurs in proximity to Wittgenstein's reference to Hertz's dynamical models:

> 4.0141 In the fact that there is a general rule by which the musician is able to read the symphony from the score, and that there is a rule by which one could reconstruct the symphony from the line on a gramophone record and from this again—by means of the first rule—construct the score, herein lies the internal similarity between these things which at first sight seem to be entirely different. And the rule is the law of projection which projects the symphony into the language of the musical score. It is the rule of translation of this language into the language of the gramophone record.[27]

This passage presents the following idea: given two representations of the same fact (such as a gramophone record and the corresponding score), it is not only possible to 'project' from either of them to what they represent (to play the record, say, or read the score), but also possible to project from one representation to the other (to write a score by listening to the record, or make a record by playing from the score). There are thus "laws of projection": *general rules* which carry us from one representation to another, such as the "rule of translation" from the language of musical notation to the language of gramophone records.[28] Again, this comparison between linguistic and non-linguistic representation is not metaphorical, "A gramophone record, the musical idea, the written notes, and the sound-waves, all stand to one another in the same internal relation of depicting that holds between language and the world" (4.014). Indeed, at the heart of the Tractarian picture-theory is the claim that the use of a propositional sign is like the use of any picture or model—in all such cases we use an arrangement of perceptible things as a projection of a possible situation.[29] On this view we can understand the sense of a proposition in much the same way as we can understand the sense of an arrangement of physical objects:

> 3.1431 The essence of a propositional sign is very clearly seen if we imagine one composed of spatial objects (such as tables, chairs, and books) instead of written signs.
>
> Then the spatial arrangement of these things will express the sense of the proposition.

We can now connect the notion of projection with the notion of multiplicity. At 4.03, Wittgenstein claims that a proposition must be "*essentially* connected" with the situation it depicts, and that this essential connection "is precisely that it is its logical picture." The notion of multiplicity is then introduced at 4.04 as an aspect of this essential connection—proposition and situation *must* have the same multiplicity. According to 4.03, both an ordinary proposition and its completely analyzed counterpart are only propositions by dint of being logical pictures of what they represent. The discussion of projection as applied to the gramophone record and the musical score at 4.0141 can thus be carried over: just as there is a general rule by which we can translate between the record and the score, so there is a general rule by which we can translate between the ordinary proposition and the fully analyzed proposition. Herein lies the "internal similarity between these things which at first sight seem to be entirely different."

The results of this section can be summed up as the drawing together of the notions of *logical picture*, *projection* and *multiplicity*. For a proposition to represent a situation is for it to be a logical picture of that situation; for a proposition to be a logical picture is for there to be a method of projection that employs its essential connection with that situation; and an aspect of this essential connection is that the proposition must have the same multiplicity as the situation it depicts. Importantly, this is as true for the fully analyzed proposition as it is for the ordinary proposition: both are logical pictures of the situation they represent, both can be used as a projection of that situation, and all three (the ordinary proposition, the fully analyzed proposition, and the situation itself) must have the same multiplicity. However, we do not yet have an understanding of what this 'multiplicity' is; we do not yet have a concrete sense of what is shared by the ordinary proposition and corresponding fully analyzed proposition such that—despite their evident differences—they both have "as much that is distinguishable" as in the situation they represent. In order to make progress on this front, we should follow Wittgenstein's prompt and turn to a consideration of Hertz's dynamical models. As will become evident, this will require an overview of *Principles* more broadly.[30]

3. *Principles of Mechanics*

The publication of *Principles* was, in a sense, the last thing that Hertz achieved. Hertz died while *Principles* was in press when he was just thirty-six years old, without having shown the manuscript to another soul.[31] Following its publication, *Principles* was met with both enthusiasm and a sense of confusion—despite its elegance and sophistication, Hertz's work seemed uncharacteristically speculative and disappointingly implausible. In *Principles*, all mechanical systems are described in terms of collections of material points with some number of connections between them. Hertz's material points are unusual, however, in that they are constituted by '*Massenteilchen*' (literally, small-mass-parts): particles which are infinitely smaller than material points, even if those material points are already infinitely small (see §§3–5).[32] Besides these strange entities, in order to avoid relying on primitive notions of force or energy Hertz also employed the notion of *hidden masses*, “motion and mass which differ from the visible ones not in themselves but in relation to us and our usual means of perception” (Hertz 1899, 25). If Hertz is interpreted as relying on a speculative ontology, however, his project can appear hopeless. In Mach’s words, working out the details of such an ontology seems to oblige one “to resort, even in simplest cases, to fantastic and even frequently questionable fictions.”[33] Similar worries about Hertz’s project were also expressed by Helmholtz and Boltzmann, and this kind of dissatisfaction with *Principles* persists up to the present day.[34]

However, there are reasons to be wary of this kind of ontological interpretation. Most immediately, these reasons stem from Hertz’s own picture-theory of representation, presented at the beginning of the lengthy philosophical introduction to *Principles*:[35]

> In endeavoring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves inner simulacra [*innere Scheinbilder*] or symbols of external objects; and the form which we give them is such that the necessary consequents of the pictures [*Bilder*] in thought are always the pictures of the necessary consequents in nature of the things pictured. . . The pictures which we here speak of are our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above-mentioned requirement. (Hertz 1899, 1)[36]

Hertz’s picture-theory is notably austere—the sole requirement on a picture is that its consequences represent the consequences of what it pictures. What Hertz strenuously emphasizes is that, on his view, the representative content of a theory does not go any further than this, “we do not know, nor have we any means of knowing, whether our conceptions of things are in conformity with them in any other than this *one* fundamental respect” (Hertz 1899, 2). Importantly, a number of commentators have argued that Hertz’s picture-theory, along with other central aspects of *Principles*, indicates that Hertz was not relying on a speculative ontology.[37] In the remainder of this section I pursue a similar interpretive strategy, focusing in particular on the central role of Hertz's dynamical models.

The customary formulation of mechanics is typically regarded as encapsulated in Newton's three laws of motion. In contrast, Hertz claims to have distilled the core empirical content of classical mechanics into a single "fundamental law":

> Every free system persists in its state of rest or of uniform motion in a straightest path. (§309)

The possibility of describing all mechanical phenomena with this fundamental law stems from the rich notion of a straightest path. In brief, *Principles* provides the resources to construct a *configuration space* representation for an arbitrary mechanical system.[38] Each such configuration space has a certain number of dimensions and a certain geometrical structure so that the path traced out by a single point represents all the mechanical properties of the system. This path through configuration space is the straightest path of Hertz's fundamental law.

Using ordinary rectangular coordinates, a mechanical system can be described in terms of the *3n* position coordinates of its *n* material points (§13) and connections between the points can be described by writing down "equations of condition" of a canonical form (§128). Rather than using *3n* rectangular coordinates, however, a system can also be described in terms of *r* general coordinates, "as long as we agree to associate continuously a given value-system of these coordinates with a given position of the system, and conversely" (§13). General coordinates are particularly useful within Hertz's framework because they can be used to incorporate information regarding the system's connections. For a large class of systems—namely, *holonomous* systems (§123)—a judicious choice of general coordinates leads to a particularly perspicuous configuration space representation of the system (see §197 ff.).[39] In particular, if *3n*

- r is equal to the number of connections then no explicit equations of condition are needed (see §129). In this case, the number of general coordinates is equal to the number of the system's *degrees of freedom*: the number of independent variables that characterize the system (see §§134–36).

As a simple example, consider a dumbbell system: two material points with a single rigid connection between them. If the locations of the two points are given by rectangular coordinates then the corresponding configuration space has six dimensions. Note that in the absence of the rigid connection the six coordinates would all be independent—any one of them could change without affecting the others. In that case the system would have six degrees of freedom and every point in the six-dimensional configuration space would represent a possible position of the system. However, the presence of the rigid connection prevents the coordinates from being entirely independent of each other, and hence many points in configuration space represent impossible positions in the sense that they would represent the connection being broken.[40] The rigid connection thus determines a 'surface' of possible positions within configuration space, and Hertz's fundamental law asserts that the motion of the system will trace out a straightest path (i.e. a path of minimum curvature) along this surface.

So far we have described this system using the rectangular coordinates of the two points and a single equation of condition, with the latter specifying the interdependence of the former and hence which positions of the system are possible positions.[41] The number of degrees of freedom of a system is equal to the number of its coordinates minus the number of its equations of condition (§135); hence our dumbbell system has five degrees of freedom. By using specially adapted general coordinates instead of ordinary rectangular coordinates, however, we can encapsulate the degrees of freedom of the system more directly. Let the three position coordinates (X, Y, Z) determine the system's center of mass, and let the two angles (θ, φ)

determine its orientation. Note that, unlike the six rectangular coordinates, these five general coordinates can all vary independently of each other. Hence *X, Y, Z, θ*, and *φ* constitute a set of specially adapted general coordinates that capture the system's five degrees of freedom, and no further equations of condition are needed. This also leads to a direct characterization of the system's surface of possible positions: the general coordinates are the curvilinear coordinates of an *r*-dimensional space, and the *3n*-dimensional embedding space has disappeared from view.

Hertz also requires that the geometry of configuration space be specially adapted to the particular mechanical system under consideration. More specifically, the determination of distances and angles—the *metric structure* of configuration space—is tied to the mass-distribution of the system. Indeed, it is precisely for the sake of imbuing configuration space with this geometrical structure that Hertz introduces the notion of *Massenteilchen.*[42] In ordinary Euclidean space, the distance between two points is determined by the familiar Pythagorean metric: $ds^2 = dx^2 + dy^2 + dz^2$. To transpose this geometrical structure into configuration space, we would simply need to generalize the Pythagorean metric to a higher number of dimensions. For a system of *n* material points, its *3n*-dimensional configuration space would then have a metric of the form: $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + \ldots + dx_{3n-2}^2 + dx_{3n-1}^2 + dx_{3n}^2$. Or more compactly:

$$ds^2 = \sum_{i=1}^{3n} dx_i^2$$

However, this metric is insufficient to do the work that Hertz requires.[43] Thus Hertz derives a more exotic metric structure, beginning by first defining the "magnitude of the displacement of a system" as follows:

> The magnitude of the displacement of a system is the quadratic mean value of the magnitudes of the displacements of all its *Massenteilchen*. (§29)

Note here the reference to *Massenteilchen*. If this definition had referred to material points, this would have resulted in configuration space having a Pythagorean metric. However, calculating the displacements of the *Massenteilchen* instead of the material points "weights" the expression for the magnitude of the displacement of a system, so that the more massive points contribute more to the displacement. Using this definition, Hertz derives a metric for configuration space of the following form (see §55):[44]

$$mds^2 = \sum_{i=1}^{3n} m_i dx_i^2$$

Here, m is the total mass of the system (equal to the sum of the masses of the material points) and the m_i are defined so that the mass of the μ-th material point is proportional to $m_{3\mu-2} + m_{3\mu-1} + m_{3\mu}$. It is worth emphasizing the importance of this metric structure: Hertz is only in a position to describe all mechanical systems with a single fundamental law because the geometrical structure of configuration space incorporates information regarding the spatial distribution of the masses making up a system. The key result is that the total kinetic energy of the system can be represented by the kinetic energy of a single point in configuration space.[45]

To sum up: in the general case, when describing a system using $3n$ rectangular coordinates and i equations of condition, the possible positions of the system will constitute a 'surface' of $3n - i$ dimensions embedded within a larger $3n$-dimensional configuration space. It is often more useful, however, to describe a system in terms of r general coordinates. For holonomous systems, a particularly perspicuous representation is possible in which $r = 3n - i$. We can then arrive immediately at the surface of possible positions: the general coordinates characterize a curved space with the same number of dimensions as the system's degrees of freedom. Whichever representation we use, the metric structure of configuration space incorporates information about the mass-distribution of the system and the motion of the system traces out a straightest path along the relevant surface of possible positions.

We can now turn to Hertz's dynamical models. These are defined as follows:

> A material system is said to be a dynamical model of a second system when the connections of the first can be expressed by such coordinates as to satisfy the following conditions:
>
> (1) That the number of coordinates of the first system is equal to the number of the second.
>
> (2) That with a suitable arrangement of the coordinates for both systems the same equations of condition exist.
>
> (3) That by this arrangement of the coordinates the expression for the magnitude of a displacement agrees in both systems. (§418)

Hertz points out that whether one system is a dynamical model of another "is independent of the choice of the coordinates of one or the other system, although it is only clearly exhibited by a particular choice of coordinates" (§420). It will be evident that the perspicuous representation of a holonomous system described above exemplifies just such a choice of coordinates; one which clearly exhibits the fact that two systems are "dynamical similar" (§419). In these coordinates, condition (1) will be satisfied just in case both systems have the same number of degrees of freedom, and condition (2) will be trivially satisfied through the fact that both systems have no equations of condition. The only thing left to check, then, is whether both systems have the same "expression for the magnitude of a displacement," that is, whether their configuration spaces come equipped with the same metric.[46]

We can gloss the relation of dynamical similarity in the following way—two systems are dynamical models of one another just in case they have the same *number* and *type* of degrees of freedom. In the case of a perspicuous representation of a holonomous system, the number and type of the system's degrees of freedom are reflected in the *dimensionality* and *geometry* of its configuration space respectively. Note that systems which have the same number and type of degrees of freedom can be constituted in various different ways. For example, because a simple pendulum, a mass on a spring, and a vibrating string can all be modeled as simple harmonic oscillators with a single degree of freedom, they can all be given identical configuration space representations. Hence systems which vary widely in their ontological constitution can nevertheless be dynamically similar. Indeed:

> An infinite number of systems, quite different physically, can be models of one and the same system. Any given system is a model of an infinite number of totally different systems. (§421)

For the purpose of representing a system's motion, all that is needed is a dynamical model of that system, one which may be "much simpler than the system whose motion it represents" (§425). Hertz makes clear that capturing the number and type of a system's degrees of freedom is sufficient to capture its mechanical properties, and that any further details will often be irrelevant (see §§327–30).

From what has been said so far, it is not yet clear how Hertz can account for all the phenomena that fall within the remit of classical mechanics. Hertz's fundamental law only applies to free systems, and hence fails to apply to systems that are *not* free, such as systems acted on by forces. This brings us to the most notorious features of Hertz's framework: the notion of hidden masses. In order to accommodate systems acted on by forces, Hertz re-defines a force as the effect one system has on another when the two are coupled together (see §455). Note that if one coupled system is 'hidden' then what is observable is a *partial* system that seems to violate the fundamental law. From Hertz's perspective any apparently unfree system is regarded as a partial system, so that every *complete* system (including any hidden masses) still follows the straightest path in its configuration space (see §429). Hertz goes on to show that every system is dynamically similar to a large family of systems which include hidden masses.[47] By hypothesis, the hidden nature of those masses means that there is no way to determine which member of this family is the true representation of the target phenomenon. Indeed, from the perspective provided by Hertz's framework, there is simply nothing further to learn about a mechanical system than what can be gleaned from a dynamical model of that system. Once the hypothesis of hidden masses is accepted, we have "no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone, that the one set of systems are models of the other" (§427).

Dynamical models thus play a central role in Hertz's framework and are intimately connected with his overarching picture-theory of representation. As Hertz's remarks following the introduction of dynamical models make clear, it is through the notion of dynamical models that he applies his fundamental requirement on pictures in general to the pictures provided by classical mechanics in particular:

> The relation of a dynamical model to the system of which it is regarded as the model, is precisely the same as the relation of the pictures which our mind forms of things to the things themselves. . . . The agreement between mind and nature may therefore be likened to the agreement between two systems which are models of one another. (§428)

Hertz's framework thus highlights the way in which mechanical descriptions of phenomena abstract away from ontological details. This is why Hertz's analytical framework is designed to capture the degrees of freedom of mechanical systems, not their ontological constitution.

I have used the ontologically oriented and logically oriented distinction to characterize contrasting interpretations of the *Tractatus*, but it should be evident that this distinction can also be used to characterize contrasting interpretations of *Principles*. On an ontologically oriented interpretation, Hertz's *Massenteilchen* would be an unfamiliar kind of fundamental (or metaphysical) particle, and the hypothesis of hidden masses would be a bold ontological gambit. On such a view, the existence of *Massenteilchen* and hidden masses would be independent of our mechanical descriptions, and appealing to the features of this ontology would provide a kind of explanation of ordinary mechanical phenomena. In contrast, on a logically oriented interpretation—the kind of interpretation defended here—the central motivation to talk in terms of *Massenteilchen* and hidden masses is to capture the essential content of mechanical descriptions, i.e. the degrees of freedom of mechanical systems. Given a particular mechanical phenomenon, we are free to analyze it into a connected system of material points, introducing hidden masses as needed. The relative masses of these material points (both hidden and visible) is what then determines the relative numbers of *Massenteilchen* occupying those locations at those times. On a logically oriented view, *Massenteilchen* and hidden masses are introduced precisely to allow for this kind of uniform analysis of mechanical phenomena. Furthermore, the features of this 'ontology' do not provide any kind of explanation of ordinary mechanical phenomena.

It will be worthwhile to note how this distinction between ontologically and logically oriented interpretations has manifested in the existing attempts to interpret Hertz's influence on the *Tractatus*. Two contrasting examples are due to Gerd Graßhoff and Sara Bizarro. Graßhoff's interpretation lies firmly in the ontologically oriented camp: on Graßhoff's view, the crucial aspect of *Principles* which influenced Wittgenstein was Hertz's *metaphysics*, "With a full grasp of its metaphysical content, Wittgenstein used [*Principles*] as the foundation for the philosophical architecture which is then built in close contention with the logical theory proposed by Russell and Frege."[48] Graßhoff claims that, although Wittgenstein did not rely on the correctness of Hertz's theory, he nevertheless thought that *some* such theory would reveal the ultimate nature of simple objects. Indeed, citing remarks in the *Notebooks* and *Prototractatus*, Graßhoff argues that Wittgenstein had Hertz's material points in mind as examples of simple objects.[49] Dovetailing with this, Graßhoff's reading of the *Tractatus* is itself manifestly ontologically oriented:

> Instead of reflecting first about language, one starts with metaphysical assumptions about simple objects and their combination in a state of affairs. . . Whether an elementary proposition matches a state of affairs is not a question of convention, since elementary propositions are true or false by virtue of their correspondence to a state of affairs. The comparison assumes the correlation between simple names and simple objects; otherwise a proposition would be senseless. At the very heart of Wittgenstein's conception lies the theory of simple objects.[50]

Hence Graßhoff takes an ontologically oriented view of both *Principles* and the *Tractatus*, and sees the influence of the one on the other in precisely such terms. But Graßhoff does not have much to say about Wittgenstein's specific reference to dynamical models at 4.04, mentioning it only as evidence that Wittgenstein read further than Hertz's introduction.[51]

Bizarro takes a very different line of interpretation to Graßhoff, though she agrees that Tractarian simple objects can best be understood via a study of *Principles.*[52] She notes, however, that the relevant parallel in *Principles* is not Hertz's notion of a material point but rather his *Massenteilchen*.[53] More importantly, she claims that "Hertz might have misled his reader to thinking that these objects have to be interpreted as physical entities."[54] Bizarro urges that Hertz's goals in *Principles* can only be properly understood through an appreciation of Hertz's picture-theory,[55] and argues that "Hertz is making an enormous effort to create a foundation for the science of mechanics that does *not* postulate *anything* about things in themselves."[56] Dovetailing with this, Bizarro regards Wittgenstein as influenced precisely by Hertz's way of circumventing questions concerning the ultimate constituents of mechanical systems. On Bizarro's view, neither Hertz nor Wittgenstein "*have* to make any claims whatsoever about the nature of reality."[57] Thus, where Graßhoff takes an ontologically oriented approach to *Principles* and the *Tractatus*, Bizarro takes a contrasting logically oriented approach to both texts. Nevertheless, like Graßhoff, Bizarro does not provide a discussion of Wittgenstein's specific reference to dynamical models at 4.04.

There is at least one commentator, however, who does examine Hertz's notion of a dynamical model, though still stops short of providing a detailed interpretation of 4.04. David Hyder occupies a middle ground between Graßhoff and Bizarro, combining an ontologically oriented approach to the *Tractatus* with a broadly logically oriented approach to *Principles*. Hyder's adoption of an ontologically oriented reading of the *Tractatus* is particularly explicit:

> The elementary proposition is a picture in that each of its elements is correlated with an element of reality, and in that it uses its own structural arrangement to replicate the structure of the fact it depicts. The complex proposition pictures as well, but it does so by means of the elementary propositions.[58]

On Hyder's interpretation, Wittgenstein wants to claim that "facts have structures that *derive* from the distinct types of things of which they are composed," and "the existence of high-level languages *depends*. . . on the primitive language's capturing a multiplicity of relations among the elementary facts" (emphasis mine).[59] Hence, according to Hyder's reading (as with Graßhoff's), the Tractarian ontology of simple objects plays a primary role. Nevertheless, Hyder's approach to Hertz is by and large logically oriented, particularly with regard to his discussion of dynamical models. Hyder makes clear that a dynamical model captures the degrees of freedom of the system it represents, *not* its ontological constitution:

> In *The Principles of Mechanics*, Hertz introduces the notion of a "dynamical model," which is a physical system standing in a particular relation to another. The most important aspect of that relation for the moment is given by Hertz's requirement that both systems have the same degree of freedom, i.e. that each have the same number of free variables, such that the values of the two systems of variables may be mapped bijectively. The class of systems standing in this transitive and symmetric relation will be vast, for there is no requirement that the actual physical components of the two systems be equal in number, nor indeed that the motions described by the one resemble those of the other: the one could be a system of strings and pulleys, the other a system of fluids and pipes. All that matters to the depictive relation is that we be able to uniquely correlate each state (*Lage*) of the one with a state of the other, and vice versa.[60]

In this way, Hyder makes clear that Hertz is able to circumvent questions concerning the ultimate constituents of mechanical systems. Indeed, Hyder writes that Hertz's dynamical models "are purely mathematical constructs possessed of only so much complexity (literally, mathematical multiplicity) as is necessary to exhaustively describe observable features of the systems they model."[61] However, Hyder does not draw on *this* notion of multiplicity in the service of interpreting 4.04 (let alone in the service of interpreting the *Tractatus* more broadly). Although Hyder does offer a detailed account of the multiplicity of elementary propositions and Wittgenstein's notion of logical space,[62] that account sits within his manifestly ontologically oriented interpretation of the *Tractatus*,[63] thus moves in the opposite direction to the logically oriented notion of multiplicity which Hyder recognizes in Hertz's dynamical models. Given that 4.04 refers precisely to dynamical models, one might wonder why Hyder never considers the possibility of a logically oriented interpretation of Hertz's influence on the *Tractatus*.[64] At any rate, in the next section this is the task to which I will turn.

To bring the current section to a close, the main reasons to favor a logically oriented interpretation of *Principles* on its own terms can be summarized as follows. The first stems from the intimate relationships between Hertz's hypothesis of hidden masses, his picture-theory of representation, and the role of dynamical models. The hypothesis of hidden masses rules out knowledge of fundamental ontological structure, and limits what we can learn about a mechanical system to the information conveyed by a dynamical model. Thus it is through the notion of a dynamical model that Hertz applies his fundamental requirement on pictures in general to the pictures provided by classical mechanics in particular—the relation of a dynamical model to the system it represents "is precisely the same as the relation of the pictures which our mind forms of things to the things themselves" (§428). A second reason to favor a logically oriented interpretation of Principles emerges in Hertz's original motivation to introduce *Massenteilchen*. As Jesper Lützen has compellingly argued, Hertz introduced *Massenteilchen* in order to derive the appropriate equation for the displacement of a system (equivalently, the appropriate metric structure for configuration space).[65] If *Massenteilchen* are interpreted as a strange kind of fundamental particle, this motivation appears wholly inadequate. On the other hand, if *Massenteilchen* are interpreted as an analytic device that allows for a uniform analysis of mechanical systems, then such a motivation is just what one might expect. A third and final reason to favor a logically oriented interpretation of *Principles* is that Hertz makes clear that he is engaged in a task of *clarification*; that his aim in *Principles* is to distill the essential content of classical mechanics from its customary representation. Hertz's succinct statement of his ambitions at the end of his preface is thus entirely consonant with a logically oriented interpretation of his work:

> As to the details I have nothing to bring forward which is new or which could not have been gleaned from many books. What I hope is new, and to this alone I attach value, is

> the arrangement and collocation of the whole—the logical or, if one wants, the philosophical aspect of the matter [*die logische, oder, wenn man will, die philosophische Seite des Gegenstandes*]. According as it marks an advance in this direction or not, my work will attain or fail of its object. (Hertz 1899, xxiv)

By providing a uniform method for representing a mechanical system's degrees of freedom, Hertz's framework thereby provides a uniform method for displaying the essential content of ordinary mechanical descriptions. With this in view, we can now begin to interpret Wittgenstein's reference to dynamical models at 4.04. Recall:

> 4.04 There must be just as much that is distinguishable in a proposition as there is in the situation that it represents.
>
> The two must possess the same logical (mathematical) multiplicity. (Compare Hertz's *Mechanics* on dynamical models.)

In Hertz's context, the 'multiplicity' that is shared by the ordinary mechanical description and a dynamical model is the number and type of the system's degrees of freedom.[66] This multiplicity is present at least implicitly, perhaps obscurely, in an ordinary mechanical description, but is made explicit by focusing on the relation of dynamical similarity. At the end of the last section I noted that we did not yet have an understanding of what is shared by the ordinary proposition and corresponding fully analyzed proposition such that they both have the same multiplicity. We have now arrived at a mechanical analogue for this multiplicity: the number and type of a system's degrees of freedom. The next question we are faced with, then, is what corresponds to the notion of degrees of freedom in the *Tractatus*.

4. The Multiplicity of a Proposition

Although it is possible to identify the multiplicity of a mechanical system concretely via its degrees of freedom, it is harder to identify the multiplicity of a proposition in a similarly concrete way.[67] However, we can make progress on this front by turning to the remarks following 4.04. In particular, at 4.0411 Wittgenstein discusses the way in which certain variants of the generality notation would fail to be adequate because they *lack* the necessary multiplicity:

> 4.0411 If, for example, we wanted to express what we now write as '$\forall x(fx)$'[68] by putting an affix in front of '*fx*'—for instance by writing 'Gen.*fx*'—it would not be adequate: we should not know what was being generalised. If we wanted to signalize it with an affix '$_g$'—for instance by writing '$f(x_g)$'—that would not be adequate either: we should not know the scope of the generality-sign.
>
> If we were to try to do it by introducing a mark into the argument places—for instance by writing '*(G, G).F (G, G)*'—it would not be adequate: we should not be able to establish the identity of the variables. And so on.
>
> All these modes of signifying are inadequate because they lack the necessary mathematical multiplicity.

To illustrate the kinds of problems that Wittgenstein identifies in these variant notations, let us take the second variant as an example. Although our standard generality notation can distinguish between propositions such as $\forall x(fx \supset p)$ and $\forall x(fx) \supset p$, using the second variant notation both propositions would be written as $f(x_g) \supset p$. Hence, as Wittgenstein points out, the variant notation is inadequate because it fails to mark the scope of the generality-sign. The other two variant notations face similarly immediate problems: we can't replace '$\forall x(fx)$' with 'Gen.*fx*' because we need to be able to identify the bound variable, and we can't replace '$\forall$x(*f x*)' with '*(G, G).F (G, G)*' because we need to be able to distinguish different variables when one quantifier occurs within the scope of another. Given that range of examples, however, it seems that we might as well regard *any* essential feature of the notation as falling under the heading 'multiplicity.' In the case of an adequate generality notation, 4.0411 helps to specify what these essential features are: any adequate generality notation needs to be able to identify bound variables, determine a quantifier's scope, and allow for one quantifier to occur within the scope of another.[69]

Indeed, elsewhere Wittgenstein explicitly distinguishes between the essential and merely accidental features of a proposition (or symbol):

> 3.34 A proposition possesses essential and accidental features.
>
> Accidental features are those that result from the particular way in which the proposition sign is produced. Essential features are those without which the proposition could not express its sense.
>
> 3.341 So what is essential in a proposition is what all propositions that can express the same sense have in common.
>
> And similarly, in general, what is essential in a symbol is what all symbols that can serve the same purpose have in common.

Equivalent (and hence equally adequate) notations have different features, and some of the features that are needed in one notation are not needed in another.[70] But such features are accidental—they result from "the particular way in which the proposition sign is produced." What is essential, by contrast, is what all adequate notations *have in common*. The upshot of the discussion of the variant generality notations in 4.0411 is that 'multiplicity' encompasses all the features of a notation that are necessary for it to do the work it purports to do. This makes the identification of such features a difficult task, and on this point Wittgenstein's reference to Hertz is particularly helpful. Within the limited scope of *Principles*, what all adequate notations have in common are the resources to represent the number and type of a mechanical system's degrees of freedom—that is the essential content of a mechanical description. Within the much broader scope of the *Tractatus*, however, what all adequate notations have in common are the resources to represent any situation *at all*. In aiming to identify the essential features of propositions *tout court*, our task becomes, in the words of 4.5: "to give a description of the propositions of *any* sign language *whatsoever* in such a way that every possible sense can be expressed by a symbol satisfying the description, and every symbol satisfying the description can give a sense. . . "[71] Here, then, we have arrived at the central Tractarian notion of the *general propositional form*.

The general propositional form purports to characterize a procedure for the construction of propositions rich enough to accommodate any proposition with sense. The logical resources that the general propositional form has available most obviously include the construction of propositions as truth-functions of elementary propositions (and truth-functions of propositions that are themselves truth-functions of elementary propositions), employing iterated applications of Wittgenstein's N operator (introduced at 5.502) to capture the familiar logical operations. However, various commentators have argued that the general propositional form also has available substantially richer logical resources than this.[72] In particular, at the level of elementary propositions these resources include the forms of elementary propositions and the forms of names of objects.[73] This allows for cross-referencing relations among different propositions through the replacement of multiple occurrences of a given name with a single variable, and hence sufficient resources to represent multiply quantified propositions.

Recall that, according to the *Tractatus*, propositions are *logical pictures* of what they represent: all propositions that can represent the world correctly or incorrectly stand in logical relations with one another. Here we find an overarching motivation to recognize all propositions with sense as instances of the general propositional form. As we saw in section two, Wittgenstein claims that pictures in general, and propositions in particular, depict reality by representing a possibility of the existence and non-existence of states of affairs (2.11, 2.201, 4.1). Writing a proposition as a truth-function of elementary propositions shows which truth-possibilities of elementary propositions the proposition agrees and disagrees with, hence which states of affairs are asserted to obtain.[74] Implication relations between ordinary propositions can then be analyzed as follows: if the truth-possibilities of elementary propositions with which a given proposition agrees include within them the truth-possibilities with which another proposition agrees, then the first proposition follows from (is implied by) the second. Wittgenstein describes this case by saying that the sense of the second proposition is *contained* in the sense of the first (5.122). Other logical relationships can be accommodated in a similar fashion: if the truth-possibilities with which one proposition expresses agreement are also the truth-possibilities with which a second proposition expresses disagreement, then the truth of either proposition implies the falsity of the other (their senses *exclude* each other), and so on.[75] Tractarian analysis thus employs the logical resources made available by the general propositional form to capture the logical relationships among colloquial propositions in terms of sense inclusion and exclusion; in terms of agreement and disagreement with truth-possibilities of elementary propositions. In this way, the complete analysis of propositions makes their logical relationships explicit.

The discussion of the variants of the generality notation at 4.0411 indicates the extent of the resources that the general propositional form needs in order to accommodate all propositions with sense. This is a characterization of the logical resources that any adequate language must at least tacitly appeal to. In a similar fashion, Hertz's dynamical models provide a characterization of the resources that any adequate *formulation of mechanics* must at least tacitly appeal to. We have thus found a Tractarian analogy with a mechanical system's degrees of freedom. In Hertz's context, what all mechanical descriptions of a given system must have in common with one another are the same degrees of freedom; these are the essential features of a mechanical description. In Wittgenstein's context, what all propositions that express the same sense must have in common with one another are the same set of logical relationships with other propositions; these are the essential features of a proposition.[76]

5. Conclusion

Both Hertz and Wittgenstein introduce unfamiliar entities, whether in the form of *Massenteilchen* and hidden masses or in the form of simple objects. One hope of this paper is that such a procedure should no longer appear surprisingly speculative. In Hertz's case, the simplicity and brevity that can be attained in the description of mechanical systems serves as a major motivation for approaching mechanical problems in an unusual way:

> we are bound to answer the question how a new, unusual, and comprehensive mode of expression justifies itself, and what advantages we expect from using it. In answering this question we specify as the first advantage that it enables us to render the most general and comprehensive statements with great simplicity and brevity. In fact, propositions relating to whole systems do not require more words or more ideas than are usually employed in referring to a single point. (Hertz 1899, 30–31)

This helps to illustrate a central feature of a logically oriented interpretation of the *Tractatus*. On such an interpretation, Wittgenstein's motivation to introduce names of simple objects is to provide a uniform method for capturing the logical relations among propositions, so that all propositions with sense can be recognized as instances of the general propositional form. In a parallel manner, Hertz provides a uniform method for capturing the degrees of freedom of mechanical systems, so that all mechanical phenomena can be seen to fall under the fundamental law. For both Hertz and Wittgenstein, then, the goal of analysis is to capture the essential features of ordinary descriptions.[77] Importantly, in neither case is there a need to appeal to the features of an underlying ontology.

Recall that, according to an ontologically oriented interpretation of the *Tractatus*, simple objects imbue the names in elementary propositions with meaning and the sense of non-elementary propositions is accounted for by appealing to the idea that they are truth-functions of elementary propositions. Furthermore, it is the simple objects occurring in states of affairs, independently of language and thought, which gives significance to the forms of names and the forms of elementary propositions. Here is David Pears' articulation of such a view:

> Wittgenstein saw the underlying structure of reality as a kind of grid of possible states of affairs, with objects at the nodal points, and it is the natures of the different types of objects which determine the way in which the grid is put together. . . this grid imposes a constraint on all factual languages: they can describe reality only in so far as they conform to it in their own underlying structure. So though different factual languages vary in superficial ways, they all have the same deep structure in common, the structure of the ultimate grid.[78]

On a logically oriented interpretation, by contrast, Tractarian analysis uncovers whatever forms of elementary propositions and forms of names are needed in order to capture the manifest logical relationships among ordinary propositions. On this view, elementary propositions and the names of simple objects do not have significance apart from the analysis of ordinary propositions.[79] I have argued that the notion of multiplicity in the *Tractatus* encompasses the essential features shared by propositions that express the same sense. These are precisely the features which the complete analysis of a proposition makes explicit, but such features must already be present, if tacit, in the ordinary proposition. (If they were not, the ordinary proposition would not be able to express its sense.) This has a clear parallel in *Principles*: a mechanical system's degrees of freedom must be tacit in an ordinary mechanical description of that system, otherwise it would not *be* a description of that system. A dynamical model simply stands to make a system's degrees of freedom explicit. Hertzian analysis is significantly more tractable than Tractarian analysis—where the *Tractatus* is concerned with the entirety of language, *Principles* is only concerned with the language of mechanics. But this means that the study of *Principles* can be a useful tool for studying the *Tractatus*. The particular point that I have urged here is that we can find in Hertz a procedure for capturing the essential content of an ordinary description which does not thereby specify the *fundamental ontological constitution* of what that description represents.

If the *Tractatus* does not provide metaphysical insights into the fundamental structure of reality in the way that a proponent of an ontologically oriented interpretation claims, one might wonder what the upshot of the *Tractatus* is supposed to be. On this note, recall that, for Hertz, one of the overarching goals of *Principles* is to alleviate confusions that trace back to conflicting demands on the term 'force.' In offering a unitary conception of force, however, Hertz does not thereby answer the question: what is the *essence* of force? In the passage from Hertz's introduction which so resonated with Wittgenstein, Hertz writes:

> the answer which we want is not really an answer to this question. It is not by finding out more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and thus perhaps by reducing their number. When these painful contradictions are removed, the question of essence [*die Frage dem Wesen*] will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions. (Hertz 1899, 8)

Hertz's suggestion, then, is that once a certain clarity is achieved, certain confused questions will no longer seem pressing. As already noted, it would be uncontentious to claim that this idea played an important role in Wittgenstein's later conception of the ambitions of philosophy. What is less widely appreciated, however, is the extent of this Hertzian influence already in the *Tractatus*.[80] The concern of this paper has been to interpret Wittgenstein's reference to Hertz's dynamical models at 4.04 and thereby uncover the parallels between Wittgenstein's analysis of propositions and Hertz's analysis of mechanical systems. All this, however, only makes more plausible the idea that Wittgenstein's conception of philosophical problems was deeply influenced by Hertz already in the *Tractatus*: Wittgenstein took inspiration both from the way in which Hertz provided an analytical framework for classical mechanics *and* from Hertz's conception of what providing such an analytical framework achieved.[81]

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[1] See Kjaergaard, "Hertz and Wittgenstein's Philosophy of Science," 126, and Janik, *Wittgenstein's Vienna Revisited*, 149.

[2] See McGuinness, *Approaches to Wittgenstein*, ix.

[3] For a discussion of how Hertz achieved this, see my "Hertz's *Mechanics* and a Unitary Notion of Force" (forthcoming).

[4] Translating Wittgenstein's term '*Satz*' as either 'sentence' or 'proposition' is the subject of some debate; see, for example, Black, *Companion*, 99. For simplicity, I follow the majority of commentators in using 'proposition' throughout.

[5] Besides Black's *Companion*, examples of ontologically oriented interpretations of the *Tractatus* include Griffin, *Wittgenstein's Logical Atomism*, Hacker, *Insight and Illusion*, and Pears, *The False Prison*. Examples of logically oriented interpretations include Ishiguro, "Use and Reference of Names," Rhees, *Discussions of Wittgenstein*, Diamond, "Throwing Away the Ladder," Kremer, "Contextualism and Holism," McGuinness, "Supposed Realism," Goldfarb "Das Überwinden," and Ricketts, "Analysis". For a similar division of interpretations of the *Tractatus* into two broad camps, see Kremer, "Contextualism and Holism," 107–08.

[6] Note that my deployment of the expressions 'bottom-up 'and 'top-down 'is not exactly the same as that of other commentators (such as Ricketts, "Analysis," 272) though it is closely related.

[7] I have in mind Russell, *The Philosophy of Logical Atomism*, and the various earlier texts that feed into this work.

[8] See Russell, "Knowledge by Acquaintance," 108.

[9] See Russell, "Knowledge by Acquaintance," 117.

[10] Textual support for an ontologically oriented interpretation of simple objects is most obvious in the 2.02s: “Objects make up the substance of the world” (2.021), “Objects, the unalterable, and the subsistent are one and the same” (2.027), and so on.

[11] Hacker, *Insight and Illusion*, 65–66.

[12] For some further discussion of a logically oriented interpretation of simple objects, see Ishiguro, “Use and Reference,” 21 and 45–46, Kremer, “Contextualism and Holism,” 98–99, and Ricketts “Analysis,” 275–77.

[13] Rush Rhees ’criticism of Black’s *Companion* provides a helpful statement of these contrasting approaches to the *Tractatus*; see Rhees, *Discussions*, 23.

[14] There are two references to *Principles* in the *Tractatus*, at 4.04 and 6.361. In this paper I will only be concerned with Wittgenstein’s first reference to *Principles*.

[15] Following convention, I give references to the *Tractatus* by citing the line number. Unless otherwise noted, the translation used is that of Pears and McGuinness.

[16] My thanks to Daniel Kaplan for helpful suggestions concerning how to render this in English. Note that David Hyder gives a similar translation of 4.04 as well as earlier iterations of this remark in Wittgenstein’s *Notebooks*; see Hyder, *The Mechanics of Meaning*, 133 and 143.

[17] This example is also discussed earlier, at 3.1432.

[18] See 2.0131: “A spatial object must be situated in infinite space. . . A speck in the visual field, though it need not be red, must have some color: it is, so to speak, surrounded by color-space. Notes must have some pitch, objects of the touch some degree of hardness, and so on.” See also 2.031–033.

[19] A more complicated example is the model of the car accident used in a Paris courtroom that Wittgenstein took as inspiration; see *Notebooks*, 7. Note that it is reasonable to treat ‘model ’

(*Modell*) and ‘picture ’(*Bild*) as synonyms in the context of the *Tractatus*; see 2.12, 4.01 and 4.463.

[20] Note that the items on my desk *only* provide a model of the parked vehicles insofar as one of them is situated determinately between the other two, and that further facts about these items (concerning their color, shape, size, etc.) are simply representationally inert.

[21] Here, obviously enough, the words ‘truck’, ‘car ’and ‘taxi ’each goes proxy for the relevant vehicle, while ‘*x* is parked between *y* and *z* ’conveys their linear ordering.

[22] I will not consider issues concerning propositions which are manifestly truth-functionally complex; for some relevant discussion, see Ricketts, “Pictures, Logic, and the Limits of Sense,” 80–88.

[23] More fine-grained logical relationships are possible. In particular, the correctness of one picture may give a *degree of probability* to the correctness of another (see 5.15).

[24] See also 4.1, “Propositions represent the existence and non-existence of states of affairs.”

[25] See 2.061–062. I will not attempt to enter into the controversy concerning Wittgenstein’s independence condition, and in particular his motivation to introduce this condition. For an ontologically oriented approach to this matter, see Pears, “Logical Independence;” for a logically oriented approach, see Ricketts, “Analysis”.

[26] As Rush Rhees has noted, the notion of projection can be recognized as intrinsic to the Tractarian conception of picturing from the get-go; see Rhees, *Discussions*, 39–40.

[27] Here I am quoting the translation by Ogden and Ramsey.

[28] Both the simple idea of projection and the potential complexities involved are evident in this example. Although we may be perfectly confident that the written score can indeed be

reconstructed from the gramophone record, actually carrying out such a reconstruction could prove very difficult in practice, especially in the absence of a record player!

[29] See 3.11, "We use the perceptible sign of a proposition (spoken or written, etc) as a projection of a possible situation. Thinking the sense of the proposition is the method of projection." Note that the translation of the second sentence, "*Die Projektionsmethode ist das Denken des Satz-Sinnes"*, is the subject of some discussion; see Winch, *Trying to Make Sense*, 13–14 and Rhees, *Discussions*, 39.

[30] It is noteworthy that in the earlier iteration of 4.04 that is numbered as 4.074 in the *Prototractatus*, Wittgenstein refers his reader simply to "Hertz's *Mechanics*". Hence it was presumably only during the final stages of compiling the *Tractatus* that Wittgenstein chose to emphasize dynamical models in particular.

[31] See Hertz's letter to his parents from 19 November 1893, published in Hertz, *Memoirs, Letters, Diaries*, 343.

[32] From this point onwards I will use section numbers without a further citation to refer to passages from the main body of *Principles*.

[33] Mach, *The Science of Mechanics*, 323.

[34] For a discussion and criticism of this tendency to interpret Hertz's work ontologically, see my "Mechanics without Mechanisms."

[35] Here and elsewhere I indicate deviations from the published English translation of *Principles* by giving the original German in square brackets. See Hertz, *Prinzipien*, 1894.

[36] There are evidently two notions of necessity in play here: necessity in thought and necessity in nature ('*denknotwendig* 'and '*naturnotwendig* 'respectively). Roughly, the first notion is concerned with inferential relationships, whilst the second is concerned with causal relationships.

Hertz's fundamental requirement on pictures is the requirement that these two notions come into alignment.

[37] See, for example, Cassirer, *The Problem of* Knowledge, 103 and especially 108–09; D'Agostino, "Hertz's Researches," 62; Nordmann," 'Everything Could be Different'," 160; and van Fraassen, *Scientific Representation*, 201.

[38] Putting it this way is anachronistic insofar as Hertz himself never used the expression 'configuration space'. For some relevant discussion, see Jesper Lützen, *Mechanistic Images*, 129–31 and 154–56.

[39] For some discussion of holonomous and non-holonomous systems, see Lützen, *Mechanistic Images*, 192–97. Although the inclusion of non-holonomous systems complicates Hertz's mechanics, the notion of a dynamical model is still applicable (see §422).

[40] Hertz discusses the relationship between the connections of a system and the possibility or impossibility of its positions in Book 1, Chapter IV. See in particular §§109–14.

[41] If the rigid connection has length l, the equation of condition for this system can be written:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = l^2$$

[42] For a detailed discussion of the development of the idea of *Massenteilchen* in the early drafts of *Principles*, see Lützen, *Mechanistic Images*, 146–158.

[43] See Lützen, *Mechanistic Images*, 154–56 for some discussion.

[44] When using general coordinates the metric structure of configuration space takes a somewhat different form; see §57.

[45] See Cornelius Lanczos, *The Variational Principles of Mechanics*, 17–24.

[46] A difference in metric structure would capture the difference between symmetrical and asymmetrical dumbbell systems, for example. Although both these systems have five degrees of

freedom, they would undergo different motions and would not be dynamical models of one another.

[47] This is made particularly vivid at §536.

[48] Gerd Graßhoff, "Hertzian Objects."

[49] See Graßhoff, "Hertzian Objects," 102 and 116. In particular, Graßhoff points to remarks from 20th and 21st June 1915 in Wittgenstein's *Notebooks*, and 2.0141 in the *Prototractatus*.

[50] Graßhoff, "Hertzian Objects," 95.

[51] See Graßhoff, "Hertzian Objects," 99.

[52] Bizarro begins by criticizing a sense-impression interpretation of simple objects, and divides readings of the *Tractatus* into 'objectivist' and 'subjectivist', but this does not affect the feature of her view that is most relevant for current purposes, i.e. that it lies firmly in the logically oriented camp.

[53] Hyder, *Mechanics*, 170, note 25, also criticizes Graßhoff's claim that Wittgenstein regarded Hertz's material points as examples of Tractarian simple objects, though for quite different reasons to Bizarro.

[54] Sara Bizarro, "A Hertzian Interpretation," 157.

[55] See Bizarro, "A Hertzian Interpretation," 158.

[56] Bizarro, "A Hertzian Interpretation," 159.

[57] Bizarro, "A Hertzian Interpretation," 162.

[58] Hyder, *The Mechanics of Meaning*, 130.

[59] Hyder, *The Mechanics of Meaning*, 136 and 158.

[60] Hyder, *The Mechanics of Meaning*, 148.

[61] Hyder, *The Mechanics of Meaning*, 172. Furthermore, Hyder writes that "Hertz's method deliberately leaves open the question of what would constitute a complete characterisation of a physical system, beyond, that is, its empirical adequacy."

[62] See Hyder, *The Mechanics of Meaning*, 113–151.

[63] This itself sits within Hyder's overarching argument that both *Principles* and the *Tractatus* are examples of neo-Kantian 'manifold theories.' In this vein, Hyder draws significantly more on the writings of Helmholtz than of Hertz while flagging that there is "no evidence that Wittgenstein had read Helmholtz himself" (Hyder, *The Mechanics of Meaning*, 13).

[64] Hyder does not appear to recognize the possibility of a logically oriented interpretation of the *Tractatus*—although he cites three of the four texts that I have listed as examples of ontologically oriented interpretations (in note 5, above) he cites no examples of logically oriented interpretations.

[65] See Lützen, *Mechanistic Images*, 146–58.

[66] For a more abstract discussion of the role of degrees of freedom in classical mechanics, see Erik Curiel, "Classical Mechanics," 273–77.

[67] Furthermore, in the context of the *Tractatus* we have no 'external 'perspective—no perspective outside of logic and language from which to reflect on logic and language. This is a peculiar and central problem at the heart of the *Tractatus* that has no analogue in *Principles*. Here I just mention it in passing.

[68] Note that I have substituted the more familiar notation, '∀*x(fx)*', for the notation used in the Tractatus, '*(x).fx* '

[69] See Michael Kremer, "The Multiplicity of General Propositions," 411–12.

[70] Compare, for example, the use of parentheses in Russellian notation with the absence of parentheses in Polish notation.

[71] 4.5 continues: "provided that the meanings of the names are suitably chosen." This is one of several remarks (including 2.1514–1515, and 6.124) that seems to emphasize the importance of the correlations between names and simple objects, and hence might be read as evidence for an ontologically oriented interpretation. However, the proponent of a logically oriented interpretation can appeal to 5.526: "We can describe the world completely by means of fully generalized propositions, i.e. without first correlating any name with a particular object."

[72] The adequacy of Wittgenstein's N operator has been the subject of controversy in the literature, going back at least to the dispute between Geach, "Wittgenstein's Operator N," and Fogelin, "Wittgenstein's Operator N," and picked up by Soames, "Generality, Truth Functions, and Expressive Capacity", McGray, "Wittgenstein's N Operator," and Connelly, "On Operator N."

[73] The form of a particular object is its possibilities of being related to other objects in states of affairs (2.0141), and the form of a particular state of affairs is the possibility that objects be related in that way (2.031–033). On an ontologically oriented interpretation of the *Tractatus*, the form of a name mirrors the form of the object it names, and the form of an elementary proposition mirrors the form of the state of affairs it asserts to obtain. A logically oriented interpretation also recognizes the forms of names and elementary propositions but without giving priority to forms of objects and states of affairs.

[74] Of course, actually writing down the complete analysis of a colloquial proposition is, at best, something we could only hope to aspire to (whether using Wittgenstein's N operator or

otherwise). For a discussion of the limited practical applicability of Tractarian notation, and concerns regarding quantifying over infinite domains in particular, see Connelly, *On Operator N*.

[75] Martin Pilch provides a useful discussion of the Tractarian construal of these kinds of logical relationships among propositions in "Wittgenstein's Logical Space." My thanks to an anonymous reviewer for bringing Pilch's work to my attention.

[76] See 5.141: "If p follows from q and q from p, then they are one and the same proposition." Thus two superficially different *signs* can be one and the same *symbol* (see 3.32): for instance, 'p ⊃ q 'is the same symbol as '~p ∨ q', and is expressed using Wittgenstein's N operator as 'N (N (N (p), q))'.

[77] Compare this with the use of the general propositional form canvassed by Cora Diamond in "What Can You Do with the General Propositional Form?" 190, "The kind of use that is in question here is the same as that which Wittgenstein speaks of in *Philosophical Investigations* when he mentions Indian mathematicians saying 'Look at this'. Something is put before us which enables us to see a formal similarity which we had not earlier been aware of. The [general propositional form] is meant to put before us an essential similarity in our use of signs."

[78] Pears, *The False Prison*, 6.

[79] See Ricketts, "Analysis," 275, "we have no grasp on what different forms of objects are, except via the interlocking contrasts among those forms that give different forms of elementary sentences different roles in capturing manifest logical relationships."

[80] Michael Kremer is one of the few commentators who recognizes Hertz's influence here; see "Russell's Merit," 16: "Wittgenstein encountered Hertz's ideas even before he became a student of Russell's, and it is my contention that the conception of philosophical problems and their solution that he found in Hertz was crucial to his approach to philosophy from the beginning."

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