Hertz's *Mechanics* and a Unitary Notion of Force

Joshua Eisenthal

Abstract

Heinrich Hertz dedicated the last four years of his life to a systematic reformulation of mechanics. One of the main issues that troubled Hertz in the traditional formulation was a 'logical obscurity' in the notion of *force*. However, it is unclear what this logical obscurity was, hence it is unclear how Hertz took himself to have avoided this obscurity in his own formulation of mechanics.

In this paper, I argue that a subtle ambiguity in Newton's original laws of motion led to the development of two slightly different notions of force: *Newtonian* and *Lagrangian*. I then show how Hertz employed the mathematical apparatus of differential geometry to arrive at a unitary notion of force, thus avoiding the logical obscurity that lurked in the customary representation of mechanics.

1 Introduction

In 1890 Heinrich Hertz began a grand project in the foundations of physics, aiming to systematically reformulate the theory of mechanics. Tragically, however, the next four years of labor were to prove to be the last four years of Hertz's short life. On New Years Day of 1894, Hertz died just thirty-six years old while the manuscript of his treatise was still in press. Thus the broader physics community greeted the publication of *The Principles of Mechanics Presented in a New Form* with a mixture of sadness and curiosity, wondering what it was, exactly, that Hertz had poured his last energies into.

In his introduction, Hertz explains that his motivation to reformulate mechanics stemmed from a sense of discomfort concerning a lack of clarity in the foundations of the theory:

> I have not attempted this task because mechanics has shown signs of inappropriateness in its applications, nor because it in any way conflicts with

> experience, but solely in order to rid myself of the oppressive feeling that to me its elements were not free from things obscure and unintelligible. (Hertz, 1899, 33)

In seeking to convey this issue to his readers, Hertz first discusses the elementary problem of swinging a stone in a circle, aiming to illustrate how easy it is 'to attach to the fundamental laws considerations which are quite in accordance with the usual modes of expression in mechanics and which yet are an undoubted hindrance to clear thinking' (Hertz, 1899, 5). He then offers three 'general observations' as further evidence for this lack of clarity. The first is the dissatisfaction felt in introducing the basic concepts and definitions of mechanics, and the desire 'to move rapidly over the introductory material on to examples which speak for themselves' (Hertz, 1899, 7). The second is the existence of disagreements concerning the rigor of supposedly elementary theorems in mechanics, disagreements which 'in a logically complete science, such as pure mathematics... [are] utterly inconceivable' (ibid). The final observation is the concern felt in the physics community over the 'essence' (*Wesen*) of *force*:

> Weighty evidence seems to be furnished by the statements which one hears with wearisome frequency, that the essence of force is still a mystery, that one of the chief problems of physics is the investigation of the essence of force, and so on. In the same way electricians are continually attacked as to the essence of electricity. Now, why is it that people never in this way ask what is the essence of gold, or what is the essence of velocity? Is the essence of gold better known to us than that of electricity, or the essence of velocity better than that of force? Can we by our conceptions, by our words, completely represent the essence of any thing? Certainly not. (Hertz, 1899, 7-8)

For Hertz, the fact that the essence of something is regarded as mysterious implies the existence of an underlying state of confusion. Hence he prompts his readers to consider why we might be drawn to ask such questions with regard to certain concepts but not others:

> I fancy the difference must lie in this. With the terms "velocity" and "gold" we connect a large number of relations to other terms; and between all these relations we find no contradictions which offend us. We are therefore satisfied and ask no further questions. But we have accumulated around the terms "force" and "electricity" more relations than can be completely reconciled amongst themselves. We have an obscure

> feeling of this and want to have things cleared up. Our confused wish finds expression in the confused question as to the essence of force and electricity.

One of the aims of *Principles* is thus to alleviate the confusion surrounding the notion of force so as to avoid confused questions as to its essence from arising. As Hertz famously put the matter:

> the answer which we want is not really an answer to this question [what is the essence of force?]. It is not by finding out more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and thus perhaps by reducing their number. When these painful contradictions are removed, the question as to essences will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions. (Hertz, 1899, 8)

The present paper has two related goals. First, I aim to give an account of the state of confusion that surrounded the notion of force as Hertz understood it. Second, I aim to show how Hertz derived a perspicuous notion of force in his own reformulation of mechanics so that our minds, no longer vexed, would cease to ask illegitimate questions. I begin in section 2 with Hertz's discussion of the 'hindrance to clear thinking' encountered in the case of swinging a stone in a circle—a difficulty whose roots, Hertz suggests, can be traced back to Newton's laws of motion. In seeking to understand the basis of Hertz's concerns, I turn in section 3 to consider the way in which the historical development of mechanics was understood in this period, focusing in particular on Thomson and Tait (1879). With this in hand, I argue in section 4 that there is indeed an ambiguity in the notion of force suggested by Newton's laws, and that Hertz was therefore well motivated in seeking a *unitary* notion of force which avoided such ambiguity. In section 5, I provide an overview of Hertz's reformulation of mechanics, leading to a discussion in section 6 of the notion of force that Hertz's framework makes available. Section 7 concludes.

2 Swinging a stone in a circle

As already noted, Hertz's first attempt to convey the lack of clarity in the foundations of mechanics involves a discussion of a simple example: 'We swing in a circle a stone tied to a string, and in so doing we are conscious of exerting a force upon the stone' (Hertz, 1899, 5). As Hertz notes, it is easy to confirm that in relation to the stone's circular motion the magnitude of this force is consistent with Newton's second law. Newton's third law then instructs us that an equal and opposite reaction force must act from the stone on our hand. With regard to this opposing force, 'the usual explanation is that the stone reacts upon the hand in consequence of centrifugal force, and that this centrifugal force is in fact exactly equal and opposite to the force that we exert' (Hertz, 1899, 5). It is here, however, that Hertz believes we encounter a hindrance to clear thinking:

> Now is this mode of expression permissible? Is what we call centrifugal force anything else than the inertia of the stone? Can we, without destroying the clearness of our conceptions, take the effect of inertia twice into account—firstly as mass, secondly as force? In our laws of motion, force was a cause of motion, and was present *before* the motion. Can we, without confusing our ideas, suddenly begin to speak of forces which are a consequence of motion? Can we behave as if we had already asserted anything about forces of this new kind in our laws, as if by calling them "forces" we could invest them with the properties of forces? These questions must clearly be answered in the negative. (Hertz, 1899, 5-6)

In section 4, below, I will return to this passage and offer an account of the concerns that lie in the background of Hertz's remarks. But we can begin by noting that there are good reasons to feel dissatisfied with the appeal to centrifugal force as a way to accommodate Newton's third law in this context. The expression 'centrifugal force' is normally used to describe what appears as a force in a rotating frame of reference. It is thus a (potentially misleading) way of describing how things appear to an observer undergoing circular motion. To illustrate: an insect clinging to the swinging stone would feel as if a force was pulling it off the surface, just as an observer inside a sharply accelerating vehicle would feel as if a force was pulling them backwards. For this reason, it is natural to call such forces *inertial* forces.[1] But Hertz complains: 'what now becomes of the demands of the third law, which requires a force exerted by the inert stone upon the hand, and which can only be satisfied by an actual force, not a mere name?' (Hertz, 1899, 6) Indeed, as things stand, appealing to centrifugal force is a confusing way to account for the force *on our hand* when swinging the stone. That force is not an inertial force and does not arise because our hand is undergoing circular motion (it isn't).

[1] Arnold Sommerfeld calls centrifugal and similar inertial forces 'fictitious', see Sommerfeld (1952) 59. In contemporary discussions they are more frequently called *pseudo*-forces.

After making this brief critique of such 'usual modes of expression in mechanics', Hertz gestures at what he takes to be the ultimate source of the problem—a deep-rooted tension in the foundations of the theory:

> I do not regard these as artificial difficulties wantonly raised: they are objections which press for an answer. Is not their origin to be traced back to the fundamental laws? The force spoken of in the definition and in the first two laws acts upon a body in one definite direction. The sense of the third law is that forces always connect two bodies, and are directed from the first to the second as well as from the second to the first. It seems to me that the conception of force assumed and created in us by the third law on the one hand, and the first two laws on the other hand, are slightly different. This slight difference may be enough to produce the logical obscurity of which the consequences are manifest in the above example. (Hertz, 1899, 6)

Here we have arrived at Hertz's most explicit indication of the underlying issue: Newton's third law presents a slightly different conception of force compared to the first and second laws, and it is this subtle ambiguity that is responsible for the 'logical obscurity' manifest in the case of the swinging stone.

However, Hertz's remarks failed to convince many of his readers. George Francis FitzGerald, for example, thought that Hertz had simply misunderstood Newton's third law:

> Hertz seems to consider that there is some outstanding confusion in applying the principle of equality of action and reaction, and appears to hold that by this principle the action on the body requires some reaction *in the body* whose acceleration is the effect of the force. He does not seem fully to appreciate that action and reaction are always on *different* bodies. From his consideration of this, and from a general review of our conception of force, he concludes that there is something mysterious about it, that its nature is a problem in physics, like the nature of electricity. (Hertz and Mulligan, 1994, 372)

FitzGerald's suggestion is that Hertz mistakenly thought that action-reaction force pairs act on one body, rather than on different bodies. But as should already be apparent, Hertz was perfectly aware that action-reaction force pairs act on different bodies, and that, in the case of the swinging stone, the reaction to the force exerted by our hand on the stone 'requires a force exerted by the inert stone upon the hand'

(Hertz, 1899, 6). Furthermore, FitzGerald's reading is unsatisfactory insofar as it is implausible that Hertz had such an elementary misunderstanding of Newton's third law and that *this* would have led him to spend the last four years of his life reformulating mechanics.

That being said, it is true that Hertz did not do much more than gesture at the underlying problem. Following his suggestion that the notion of force in Newton's third law may be slightly different to that in the first two laws, Hertz remarks that it is not necessary to discuss further examples. Instead, he appeals to 'general observations as evidence in support of the above-mentioned doubt' (Hertz, 1899, 6). He also appeals to the fact that similar doubts have been expressed by a number of other physicists as well: 'It is not going too far to say that this representation has never attained scientific completeness... This is also the opinion of distinguished physicists who have thought over and discussed these questions' (Hertz, 1899, 8). In all this, Hertz is only motivating the need for a logically perspicuous reformulation of mechanics, and stops short of decisively demonstrating the need for it. Uncovering the underlying basis of Hertz's concern is evidently left to us.

3 'The customary representation of mechanics'

In order to make progress in identifying Hertz's logical obscurity, we need to appreciate Hertz's understanding of the domain in which it lies—the *customary representation of mechanics*:

> By this we mean the representation, varying in detail but identical in essence, contained in almost all text-books which deal with the whole of mechanics, and in almost all courses of lectures which cover the whole of this science. This is the path by which the great army of students travel and are inducted into the mysteries of mechanics. It closely follows the course of historical development and the sequence of discoveries. Its principal stages are distinguished by the names of Archimedes, Galileo, Newton, Lagrange. (Hertz, 1899, 4)

Newton, unsurprisingly, plays an especially important role in the development of this representation of mechanics. However, Hertz notes that Newton's laws were not sufficient for the full development of this representation without the addition of d'Alembert's principle:

> [Newton's] laws contain the seed of future developments; but they do not furnish any general expression for the influence of rigid spacial connections. Here d'Alembert's principle extends the general results of statics

> to the case of motion, and closes the series of independent fundamental statements which cannot be deduced from each other. From here on everything is deductive inference. (Hertz, 1899, 5)

Hertz's brief remarks gesture at a rich and complex history leading up to the textbooks and lecture courses of his time. Following Cornelius Lanczos, it is possible to distinguish two prominent traditions in this history: a *vectorial* tradition and a *variational* tradition.[2] In the vectorial tradition, the primary objects are force vectors between point-masses. Complex systems can be regarded as arrays of such points, each pair of which has equal and opposite forces acting along the line connecting them. Classical models of the solar system exemplify a paradigm application of this tradition. By contrast, in the variational tradition the primary object is a scalar representation of some overall property of the system. Applications of 'extremal' principles, such as various formulations of the principle of least action, are the home territory of this tradition. In the remainder of this section I will reconstruct the way in which d'Alembert's principle can be seen to provide a pathway starting from a vectorial interpretation of Newton's laws of motion and leading to the variational techniques of Lagrange and others. To make clear the received understanding of this historical development by Hertz and his contemporaries, I will focus on William Thomson (later Lord Kelvin) and Peter Tait's seminal 1879 *Treatise of Natural Philosophy* (henceforth 'TNP').[3] However, given that Hertz considered the customary representation of mechanics to be 'identical in essence' in almost all the major textbooks and lecture courses of his time, nothing important should turn on the particulars of Thomson and Tait's approach.

Thomson and Tait dedicate the entirety of their first chapter to kinematics, concluding with a general discussion of the relationship between the *constraints* acting on a system and that system's *degrees of freedom*, i.e. the number of independent variables needed to characterize its motion (TNP §195). They survey a number of standard examples: a single point moving freely in space has three degrees of freedom, whereas a point constrained to move on a surface has just two degrees of freedom, and so on (TNP §196). They then arrive at a more general conception of the coordinates which jointly specify the position and motion of an arbitrary system:

> if ψ, ϕ, θ, etc., denote any number of elements, independently variable, which, when all given, fully specify its position and configuration, being

[2]See Lanczos (1962) xvii.

[3]When Hertz lists some of his sources in his preface, he writes 'I have naturally consulted the better-known text-books of general mechanics, and especially Thomson and Tait's comprehensive treatise' (Hertz, 1899, xxiv).

> of course equal in number to the degrees of freedom to move enjoyed by the system, these elements are its *co-ordinates*. (TNP §204)

In contemporary terminology, such independent variables are *general* (or *generalized*) coordinates for a given system. Thus, for example, rather than describe the configuration of a rigid body consisting of n material points with $3n$ rectangular coordinates (three to specify the position of every point), it is vastly more economical to use six general coordinates—three to describe the position of one point (the center of mass, say) and three angles to describe the orientation of the body around this point. More broadly, the use of general coordinates leads to highly abstract descriptions of mechanical systems in terms of their degrees of freedom, and the coordinates themselves do not have to have the dimensions of position.[4]

Thomson and Tait turn to Newton's laws in their second chapter, "Dynamical Laws and Principles." As kinematics had been treated as a 'subject of pure geometry' (TNP §205), there had been no need to discuss the intrinsically dynamical notions of *matter* and *force* until this point. The initial introduction of these notions, however, is very brief:

> We cannot, of course, give a definition of *Matter* which will satisfy the metaphysician, but the naturalist may be content to know matter as *that which can be perceived by the senses*, or as *that which can be acted upon by, or can exert, force*. The latter, and indeed the former also, of these definitions involves the idea of *Force*, which, in point of fact is a direct object of sense; probably of all our senses, and certainly of the "muscular sense." (TNP §207)

Here we have a somewhat vague appeal to the paradigm cases of matter (and force) manifest in our everyday interactions with material objects. Thomson and Tait then follow Newton in defining *mass* as 'quantity of matter,' which is in turn defined as the product of volume and density. They note immediately, however, that this is really a definition of density rather than mass (TNP §208). A little later they define force as 'any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line' (TNP §217). Though there is much left to be desired by such definitions,[5] we can at least recognize the intuitive notions that Thomson and Tait have in view. With *force* in particular, they evidently have in view the familiar pushes and pulls manifest in our tactile interactions with ordinary objects. Such

[4]For some further discussion of general coordinates, see North (forthcoming) §2.2.

[5]Hertz himself complains about both; see Hertz (1899) 7.

forces are typically represented ‘as velocities are, by straight lines in their directions, and of lengths proportional to their magnitudes’ (TNP §227); represented, that is, by three-dimensional vectors in ordinary Euclidean space. For want of a label, let us call forces understood in this way ‘Newtonian’.

After introducing a number of other standard dynamical notions (momentum, kinetic energy, work, etc.), Thomson and Tait then turn to Newton’s laws themselves. Newton’s own statement of the laws is as follows:[6]

> **Law I:** Every body preserves in its state of being or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.
>
> **Law II:** A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.
>
> **Law III:** To any action there is always an opposite and equal reaction; in other words, the action of two bodies upon each other are always equal and opposite in direction.

Thomson and Tait interpret the first two laws in a familiar way. Beginning with the first, they gloss its content as follows: any force-free body moves through equal spaces in equal times; and, in the reference frame of such a body, all other force-free bodies also move through equal spaces in equal times. As regards the second law, Thomson and Tait’s discussion fleshes out the vectorial aspects of Newtonian forces:

> If any force generates motion, a double force will generate double motion, and so on, whether simultaneously or successively, instantaneously, or gradually applied. And this motion, if the body was moving beforehand, is either added to the previous motion if directly conspiring with it; or is subtracted if directly opposed; or is geometrically compounded with it according to the kinematical principles already explained... (TNP §252)

The ‘quantity of motion’ altered by a force is of course the momentum of the body in question. Hence Thomson and Tait’s remarks gesture at a vectorial treatment of momenta and forces as well as a statement of the second law in terms of changes of momentum rather than mass and acceleration.

With regard to Newton’s third law, Thomson and Tait’s discussion begins in a similarly familiar way. In particular, they spell out the natural interpretation of the

[6]Translation (based on the third edition of Newton’s *Principia* from 1726) by I. Bernard Cohen and Anne Whitman, published in Newton (2004) 70-71.

action-reaction principle in terms of Newtonian forces: if one body pushes or pulls another, then the second body will pull or push the first with an equal magnitude and in the opposite direction. Thus we are prompted to bring to mind the kinds of examples that are still ubiquitous in introductory physics textbooks: 'If any one presses a stone with his finger, his finger is pressed with the same force in the opposite direction by the stone. A horse towing a boat on a canal is dragged backwards by a force equal to that which he impresses on the towing-rope forwards' (TNP §262).

It is at this juncture, in the continuation of their discussion of Newton's third law, that Thomson and Tait introduce d'Alembert's principle. In fact, they argue that d'Alembert's principle was implicitly understood by Newton as a consequence of his third law, and even claim that the germ of the 'splendid dynamical theory' of Lagrange can be derived from the Scholium appended to the third law.[7] On Thomson and Tait's reading, it is here that Newton points out that 'forces of resistance against acceleration are to be reckoned equal and opposite to the actions by which the acceleration is produced' (TNP §264). They continue:

> by the principle of superposition of forces in equilibrium, all the forces acting on points of the system form, with the reactions against acceleration, an equilibrating set of forces on the whole system. (TNP §264)

Thomson and Tait claim that this balancing of each force acting on a system with the corresponding reaction against acceleration is 'the celebrated principle first explicitly stated, and very usefully applied, by d'Alembert in 1742' (TNP §264).[8] As they go on to show, extremely general equations of motion can be derived from this principle. In order to get to this derivation, however, it is first necessary to introduce the notion of a *conservative system* as well as the *principle of virtual velocities*.

In a conservative system, 'the mutual forces between its parts always perform, or always consume, the same amount of work during any motion whatever, by which it can pass from one configuration to another' (TNP §271). The claim here is that the work done (whether positive or negative) during the motion of a conservative system is path-independent, depending only on the system's initial and final configurations. This has immediate and important consequences which Thomson and Tait begin to spell out in their ensuing remarks. In particular, in a conservative system the potential energy 'depends solely on [the system's] configuration at that instant' (TNP §274). This means that the potential energy of a conservative system can always be represented by a scalar function. Note that systems which *cannot* be assigned such

[7] TNP §263; see also §242.

[8] For a discussion of the original formulation of the principle by d'Alembert himself, see Fraser (1985)

a scalar potential function—typically, systems involving friction—often cannot be treated by the 'abstract dynamics' which Thomson and Tait are on their way to developing. Thomson and Tait then offer the following statement of the principle of virtual velocities:

> A material system, whose relative motions are unresisted by friction, is in equilibrium in any particular configuration if, and is not in equilibrium unless, the work done by the applied forces is equal to the potential energy gained, in any possible infinitely small displacement from that configuration. (TNP §289)

As Thomson and Tait note, Lagrange made this 'celebrated principle... the basis of his *Mécanique Analytique*' (TNP §289).

On the basis of this principle, Thomson and Tait's derivation of the equations of motion for an arbitrary mechanical system is the following.[9] Consider a system of n material points (with some number of connections between them) acted on by external forces. Appealing to a rectangular coordinate system, let the position of the ν-th point be given by $x_{3\nu-2}, x_{3\nu-1}, x_{3\nu}$ and let the components of the applied force acting on it be given by $F_{3\nu-2}, F_{3\nu-1}, F_{3\nu}$. Furthermore, for notational convenience, let the mass of the point be $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$.[10] The application of d'Alembert's principle then comes to the following: each component of the applied force, F_ν, is balanced by the corresponding 'reaction against acceleration,' $-m\ddot{x}_\nu$. Making use of the principle of virtual velocities, we can now apply d'Alembert's principle to the whole system in terms of the variations of the system's coordinates, δx_ν:

$$\sum_{\nu=1}^{3n} (F_\nu - m_\nu \ddot{x}_\nu)\, \delta x_\nu = 0$$

If there are m connections between the points, these can be represented by m constraint equations of the following form:

$$f_1(x_1, ..., x_{3n}) = 0 \quad \ldots \quad f_m(x_1, ..., x_{3n}) = 0$$

Taking the variations of these equations gives:

$$\frac{df_1}{dx_1}\delta x_1 + ... + \frac{df_1}{dx_{3n}}\delta x_{3n} = 0 \quad \ldots \quad \frac{df_m}{dx_1}\delta x_1 + ... + \frac{df_m}{dx_{3n}}\delta x_{3n} = 0$$

[9] Note that I have made a number of notational changes; cf. TNP §293.
[10] Here I follow Hertz (1899) §31.

We can multiply each of these equations by a Lagrange multiplier, λ, and add them all to the d'Alembertian equilibrium condition. Equating coefficients of δx_ν then gives the following $3n$ equations:

$$\lambda_1 \frac{df_1}{dx_1} + ... + \lambda_m \frac{df_m}{dx_1} + F_1 - m_1 \ddot{x}_1 = 0$$

$$\vdots$$

$$\lambda_1 \frac{df_1}{dx_{3n}} + ... + \lambda_m \frac{df_m}{dx_{3n}} + F_{3n} - m_{3n} \ddot{x}_{3n} = 0$$

These equations are sufficient to determine the m Lagrange multipliers as well as the $3n - m$ independent variables to which the x_ν are reduced by the constraint equations. Having derived such equations, Thomson and Tait declare that d'Alembert's principle comprehends 'every possible condition of every case of motion,' and that the equations of motion 'in any particular case are, as Lagrange has shown, deduced from it with great ease' (TNP §293).

In more contemporary treatments it is straightforward to use d'Alembert's principle to derive one of several variational principles which involve finding the stationary value of a scalar quantity. For example, defining the system's Lagrangian, L, as the difference between kinetic and potential energy, and requiring that all virtual displacements are zero at t_i and t_f, it is easy to show that the requirement that the total virtual work performed by the system is zero is equivalent to the following:[11]

$$\delta \int_{t_i}^{t_f} L \, dt = 0$$

This is a statement of Hamilton's Principle. In words: given fixed endpoints, the motion of a mechanical system always occurs in such a way that the integral of the Lagrangian becomes stationary for arbitrary variations of the system's configuration.

It is in this kind of way that d'Alembert's principle paved the way to a battery of powerful variational techniques, including Lagrange's equations of motion and various formulations of the principle of least action. Thus, not only was d'Alembert's principle a pivotal advance in the history of mechanics, it also provided a bridge from the vectorial approach exemplified by Thomson and Tait's interpretation of Newton's laws of motion to the 'abstract dynamics' exemplified by the variational techniques of Lagrange and others.

[11] See Lanczos (1962) 112-113 and Butterfield (2004) 72.

4 A logical obscurity

According to d'Alembert's principle, all the forces acting on a system are balanced by equal and opposite 'reactions against acceleration.' These reactions are therefore naturally regarded as themselves forces. Returning to Lanczos, we can say that d'Alembert's principle 'introduces a new force, the force of inertia, defined as the negative of the product of mass and acceleration' (Lanczos, 1962, 92). Hence d'Alembert's principle itself can be understood as the claim that '*any system of forces is in equilibrium if we add to the impressed forces the forces of inertia*' (Lanczos, 1962, 90; emphasis in original). However, here we may recall the concerns that Hertz raised in the context of swinging a stone in a circle:

> Can we, without destroying the clearness of our conceptions, take the effect of inertia twice into account—firstly as mass, secondly as force? In our laws of motion, force was a cause of motion, and was present *before* the motion. Can we, without confusing our ideas, suddenly begin to speak of forces which are a consequence of motion? Can we behave as if we had already asserted anything about forces of this new kind in our laws, as if by calling them "forces" we could invest them with the properties of forces? (Hertz, 1899, 6)

When applied to d'Alembert's principle, Hertz's concerns appear perfectly reasonable. In Newton's first law, a body continues at a constant velocity unless it is 'compelled to change its state by forces impressed' and in the second law a force is responsible for the 'change in motion'. Forces of inertia, however, arise *because* a body has changed its state of motion (i.e. because it is accelerating). Furthermore, according to d'Alembert's principle, forces are always balanced and systems are always in equilibrium. There is thus a palpable tension between the notion of a force of inertia and the conception of force articulated in Newton's first two laws of motion.

Here we have arrived at the deeper concern behind Hertz's discussion of the swinging stone. Recall that the initial worry was simply that in using the expression 'centrifugal force' we would be describing the wrong thing—we would be describing the inertial force that would be experienced by an observer undergoing circular motion (such as an insect clinging to the stone) rather than the reaction force, required by Newton's third law, that acts from the stone on the hand. But this confused use of terminology, on its own, is a relatively superficial complaint. The deeper 'hindrance to clear thinking,' made particularly vivid by d'Alembert's principle, is the introduction of the so-called inertial forces themselves. In fact, d'Alembert's principle helps to bring the importance of inertial forces to the surface, indicating that they may have just as much claim to reality as other kinds of forces. And here we can see more clearly the tension that Hertz identifies between the notion of force articulated in Newton's first two laws and the notion of force articulated in Newton's third law. For with the d'Alembertian 'reactions against acceleration' in view, it now looks like we have forces which are *causes of* accelerations, on the one hand, and forces which are *caused by* accelerations, on the other.[12]

The worry that we no longer have a clear notion of force is compounded when we consider how forces are recovered within the variational tradition considered on its own terms. In this tradition, a system is typically described using an appropriate set of general coordinates, q_ρ, and its equations of motion are derived from its Lagrangian, which is in turn characterized in terms of the system's kinetic and potential energy. A notion of force can then be recovered by considering the *generalized forces*, Q_ρ, that are derivable from the system's potential function, V:

$$Q_\rho = \frac{\partial V}{\partial q_\rho}$$

Such 'Lagrangian' forces are typically not the familiar pushes and pulls that we earlier labeled Newtonian forces. In particular, there is no need for them to be three-dimensional; they can have as many dimensions as the system's degrees of freedom. Furthermore, because general coordinates do not have to have the dimensions of position, Lagrangian forces do not even have to have the dimensions of force.[13]

At this juncture some might argue that Lagrangian forces should not be treated as fundamental. After all, it is a strength of the variational approach that it characterizes a system's degrees of freedom without delving into that system's mircophysical details. Indeed, it is widely assumed that—in a world governed by classical mechanics—all material objects could ultimately be described in terms of a plethora of Newtonian forces operating at the microscopic level.[14] However, it is important to note that no such foundational program has ever been successfully carried out. As Wilson (2013) notes, attempts to construct macroscopic bodies starting with atomic constituents runs into difficulties in specifying the special force laws which could make an array of point-masses cohere into an extended body. Furthermore, going on to specify how such a body responds under impacts, for example, can be extremely difficult, requiring modeling assumptions that plainly have 'no relationship to any structure present in real-life materials' (Wilson, 2013, 68). Indeed, this fact

[12]For some further discussion of the connections between inertial forces and d'Alembert's principle, see Lanczos (1962) 96-100.

[13]Nevertheless, $Q_\rho \delta q_\rho$ always has the dimensions of work; see Butterfield (2004) §3.3.1.B.

[14]For criticisms of such an assumption, see Wilson (2013) and Butterfield (2004).

shouldn't be particularly surprising—we have long since given up the expectation that interactions at atomic length scales are governed by classical physics. Beyond this, we have ample evidence that vectorial and variational approaches move in quite different directions. On the one hand, we know that within the scope of classical mechanics vectorial techniques can accommodate systems with friction in ways that are not possible using variational methods. (Recall that the restriction to conservative systems with a scalar potential function was necessary in order to state d'Alembert's principle in the first place.) On the other hand, we know that many of the variational principles originally developed in the context of classical physics now find a central role in both relativity and quantum mechanics.

Returning to my more immediate goal, I hope to have shown that a consideration of Thomson and Tait's canonical approach—moving from an explicitly vectorial treatment of Newton's laws, then pivoting via the scholium attached to Newton's third law in order to articulate d'Alembert's principle and thence to derive Lagrangian equations of motion—helps to flesh out Hertz's suggestion that the origin of his worries in the context of swinging a stone in a circle can be traced back to Newton's laws of motion. As Hertz specified: 'the notion of force in the third law, and the conception which the first two laws presume and suggest to us, seem to me to be slightly different, and this slight difference may be enough to produce the logical obscurity whose consequences are manifest in our example' (Hertz, 1899, 6). Hence we can appreciate Hertz's motivation to clarify the structure and scope of mechanics and, in particular, to seek out a unitary notion of force. As we turn to examine how Hertz accomplished these tasks, it will be worth bearing in mind that Hertz never regarded the lack of clarity in the customary representation as betraying a fundamental problem in the theory. Indeed, Hertz stated that 'the existing defects are only defects in form' and that 'all indistinctness and uncertainty can be avoided by suitable arrangement of definitions and notations, and by due care in the mode of expression' (Hertz, 1899, 9). Thus, if Hertz is successful in his task of clarification, he will show us how the ambiguity in the customary notion of force can be avoided.

5 Hertz's alternative approach

In his own presentation of mechanics, Hertz begins with just three primitive notions: space, time, and mass. Furthermore, in contrast to Newton's three laws of motion, Hertz distils the core empirical content of classical mechanics into a single 'fundamental law':

> **Fundamental Law.** Every free system persists in its state of rest or of

uniform motion in a straightest path. (Hertz, 1899, §309)[15]

The possibility of describing all mechanical phenomena with this fundamental law stems from the rich notion of a 'straightest path'. In brief, Hertz deploys the apparatus of differential geometry to construct *configuration space* representations of mechanical systems. Each such configuration space has a certain number of dimensions and a certain geometrical structure so that the path traced out by a single point represents all the mechanical properties of the system. This path through configuration space is the 'straightest path' of Hertz's fundamental law.[16]

Note that the fundamental law applies only to 'free' systems; roughly, systems that can be treated as isolated. As I will discuss in what follows, it was in order to accommodate systems that are *not* free that Hertz introduced the notion of *hidden masses*, a feature of *Principles* that baffled many of Hertz's readers. Importantly, this aspect of Hertz's project is also closely connected with Hertz's clarification of the notion of force. To get clear on these matters, however, we must first unpack the role that the notion of configuration space plays within Hertz's framework more generally.[17]

Consider again a system of n material points in which the position of the ν-th point is $(x_{3\nu-2}, x_{3\nu-1}, x_{3\nu})$ and its mass is $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$. Where Thomson and Tait used constraint equations of the form $f(x_1, ..., x_{3n}) = 0$, Hertz instead uses slightly more general 'equations of condition' of the form:[18]

$$\sum_{\nu=1}^{3n} x_{\iota\nu} dx_\nu = 0$$

Here, the $x_{\iota\nu}$ are continuous functions of the coordinates x_ν. As before, if there are m connections between the points there will be m equations of condition (so that the index ι runs from 1 to m). Using these rectangular coordinates, the system will have a configuration space of $3n$ dimensions. However, each connection lowers the number of the system's degrees of freedom by one and makes the region of the system's configuration space which corresponds to "breaking" that connection inaccessible. Thus each connection defines a hypersurface of $(3n - 1)$ dimensions

[15] From this point onwards, a section number without a further citation will be used to refer to passages from the main body of *Principles*.

[16] The geometry of configuration space incorporates information about the mass-distribution of the system; see Eisenthal (2018) 48-49 for some discussion.

[17] Note that this way of putting it is anachronistic insofar as Hertz himself never used the term 'configuration space.' Indeed, Hertz was keen to separate the apparatus of high-dimensional geometry from any connotations of a high-dimensional "space"—see Hertz (1899) 30.

[18] See §§125–128.

within the system's full $3n$-dimensional configuration space. The accessible region of configuration space will have a total of $3n - m$ dimensions: the intersection of all the hypersurfaces determined by the connections. Hertz's fundamental law asserts that the motion of a free system always traces out a straightest path on this curved hypersurface embedded within its $3n$-dimensional configuration space.

A more abstract and more powerful characterization of a system can be achieved by describing it in terms of its general coordinates, q_ρ.[19] Introducing '$q_{\chi\rho}$' as continuous functions of these coordinates (§130), the equations of condition can then be expressed via:

$$\sum_{\rho=1}^{r} q_{\chi\rho} dq_\rho = 0$$

For a large class of mechanical systems, the use of general coordinates leads to a particularly elegant characterization of a system's configuration space: the coordinates characterize a curved (Riemannian) space with the same number of dimensions as the system's degrees of freedom. By using these coordinates we arrive immediately at the relevant "surface" of possible motions—the embedding space of impossible motions has disappeared from view.[20]

As noted above, Hertz's fundamental law only applies to free systems and hence fails to apply to systems that are *not* free, such as systems acted on by forces. This brings us to the most notorious feature of Hertz's framework—the notion of *hidden masses*. In order to accommodate systems acted on by forces, Hertz re-defines a force as the effect one system has on another when the two are coupled together (§455). Note that if one coupled system is 'hidden' then what is observable is a *partial* system that seems to violate the fundamental law. From Hertz's perspective any apparently unfree system is regarded as a partial system, so that every *complete* system (including any hidden masses) still follows the straightest path in its configuration space.[21]

Hertz considers two classes of partial systems: *guided systems* and *systems acted on by forces*. In the latter case, two (or more) material systems form a combined system by having at least one coordinate in common. Such systems are thereby

[19]Note that, for consistency, I am using q_ρ to represent general coordinates even though Hertz himself uses p_ρ. I will also continue to use Q_ρ to represent generalized forces even though Hertz uses P_ρ.

[20]For a more detailed discussion, see Eisenthal (forthcoming) §3.

[21]'Every unfree system we conceive to be a portion of a more extended free system; from our point of view there are no unfree systems for which this assumption does not obtain.' (§429)

'coupled' to one another (§450). It is here, then, that we find Hertz's re-introduction of the notion of force:

> By a *force* we understand the independently conceived effect which one of two coupled systems, as a consequence of the fundamental law, exerts upon the motion of the other. (§455)

Let the r coordinates q_ρ and the $\boldsymbol{r}$ coordinates $\boldsymbol{q_\rho}$ describe two coupled systems, A and B, respectively, and let us arrange the indices so that the common coordinates in both systems have the same index. Hence the fact that the systems are coupled is expressed via the condition that, for at least one value of ρ:

$$\boldsymbol{q_\rho} - q_\rho = 0$$

As this holds continually, we also have:

$$d\boldsymbol{q_\rho} - dq_\rho = 0$$

Each of the coupled systems has its own equations of condition. In general coordinates, these are expressed as follows:

$$\sum_{\rho=1}^{r} q_{\chi\rho} dq_\rho = 0$$

$$\sum_{\rho=1}^{\boldsymbol{r}} \boldsymbol{q}_{\chi\rho} d\boldsymbol{q}_\rho = 0$$

We are seeking an expression for the effect that one coupled system has upon the other, and hence the force (as Hertz defines that term) that it exerts. Let us consider the force that the system B exerts on the system A.[22] For the *uncoupled* coordinates, the equations of motion of the system have the following form (§457):

$$mf_\rho + \sum_{\chi=1}^{k} q_{\chi\rho} Q_\chi = 0$$

Here, f_ρ is a generalized acceleration vector, defined as the rate of change of the system's velocity (§273). The $q_{\chi\rho}$ are the same factors that appear in the system's

[22]We could, of course, equally well consider the effect that the system A has on the system B. As Hertz stresses (§456), which of these effects should be called the 'force' and which the 'counterforce' is arbitrary.

equations of condition, whilst the Q_χ incorporate information about the curvature of the system's path. The only difference for the *coupled* coordinates turns out to be that we need to include an extra factor, Q_ρ, in these equations (§457):

$$mf_\rho + \sum_{\chi=1}^{k} q_{\chi\rho} Q_\chi - Q_\rho = 0$$

By setting $Q_\rho = 0$ for all values of ρ that are not coupled coordinates, this can be used as a more general expression for the equations of motion for the system (§459). Thus the quantities Q_ρ 'forms the analytical expression for the force which the system $\boldsymbol{q}_\rho$ exerts on the system q_ρ' (§460). Indeed: 'By this determination we place ourselves in agreement with the existing notation of mechanics; and the necessity for securing such an agreement sufficiently justifies us in choosing this particular determination out of several permissible ones' (§460).

Before turning to consider what Hertz has hereby achieved, it is important to note that the introduction of hidden masses can seem a heavy price to pay. In particular, without some reason to believe in the actual existence of such hidden masses, Hertz's approach can appear remarkably speculative. Indeed, this was the reaction of many of Hertz's contemporaries. Mach, for example, wrote that working out the details of a Hertzian system of hidden masses would oblige one 'to resort, even in simplest cases, to fantastic and even frequently questionable fictions' (Mach, 1960, p. 323). Helmholtz and Boltzmann raised similar concerns,[23] and this reaction to Hertz's project persists to the present day.[24]

However, a more satisfying interpretation of *Principles* is readily available. Within Hertz's framework, systems which have identical degrees of freedom will have identical configuration space representations even if they are otherwise quite different from one another. For example, because a simple pendulum, a mass on a spring, and a vibrating string are all simple harmonic oscillators with a single degree of freedom, they are all represented by the same configuration space. These are Hertz's *dynamical models*: two systems are dynamical models of one another just in case they

[23] In the introduction that he wrote for *Principles*, Helmholtz commented: 'Unfortunately [Hertz] has not given examples illustrating the manner in which he supposed such hypothetical mechanism to act; to explain even the simplest cases of physical forces on these lines will clearly require much scientific insight and imaginative power' (Hertz, 1899, xx). In the same vein, Boltzmann remarked, 'so long as even in the simplest cases no systems or only unduly complicated systems of hidden masses can be found that would solve the problem in the sense of Hertz's theory, the latter is only of purely academic interest' (Boltzmann, 1974, 90).

[24] See, for example, Lützen (2005) 278 and Mulligan (1998) 178. For a criticism of this tendency to interpret Hertz's hypothesis of hidden masses as surprisingly speculative, see Eisenthal (2018)

have identical configuration spaces.[25] Hence systems which vary widely in terms of their ontological constitution can nevertheless be *dynamically similar* in the relevant sense:

> An infinite number of systems, quite different physically, can be models of one and the same system. Any given system is a model of an infinite number of totally different systems. (§421)

This is crucial for making sense of Hertz's appeal to hidden masses. By hypothesis, the hidden nature of those masses means that there is no way to determine which member of a family of dynamically similar systems is the true representation of the target phenomenon. Hertz exploits this feature of his framework to make clear that, from the perspective provided by *Principles*, there is nothing further to learn about a mechanical system than what can be gathered from a dynamical model of that system. Once the hypothesis of hidden masses is accepted, we have 'no knowledge as to whether the systems which we consider in mechanics agree in any other respect with the actual systems of nature which we intend to consider, than in this alone, that the one set of systems are models of the other' (§427). With this in view, we can see that Hertz's introduction of hidden masses should not be understood as a speculative ontological gambit. Rather, Hertz recognized that in order to derive the correct equations of motion for unfree systems he only needed to be able to construct the appropriate dynamical models.[26] He thus helped himself to the analytical tools that he needed in order to be able to complete this task. With this in view, let us turn to consider the advantages of the unitary notion of force that Hertz has derived.

6 Hertz's notion of force

Hertz's notion of force adds nothing beyond the application of the fundamental law to a system of connected material points. Even so, Hertz is able to show that his notion is in accord with the notion of force in the customary representation of mechanics to a remarkable degree.

Most immediately, Hertz's forces are vector quantities (of various dimensions), both with regard to the systems that exert them and with regard to the systems that they are exerted on. All the expected compositional properties of forces are

[25] See §§418-428; for some discussion see Eisenthal (2018) 54.

[26] 'In order to determine beforehand the course of the natural motion of a material system, it is sufficient to have a model of that system. The model may be much simpler than the system whose motion it represents' (§425).

thus recovered: multiple forces can be added together to form a resultant force, and a single force can be analysed into components (§§461 ff.). Hertz also points out that various different systems can exert the same force on a given system, and also that one given system can exert the same force on various different systems. Hence a force itself (considered simply as a certain vector quantity) can be considered on its own terms, independently of the partial system that is, so to speak, responsible for it (§§464-465).

Hertz also derives sharpened up versions of Newton's three laws of motion, making explicit that the clearest and least ambiguous applications of the laws are to systems consisting of a single point.[27] Regarding the first law, Hertz first observes that a free system with no connections 'persists in its condition of rest or uniform motion in a straight path' (§382). This is just because, in the absence of any connections, the *straightest* path in configuration space is indeed the *straight* path. Hertz then notes that Newton's first law follows as a corollary of this fact concerning connection-free systems in general, because it will obviously apply to a system consisting of a single material point:

> A free material point persists in its condition of rest or uniform motion in a straight path (Galileo's Law of Inertia or Newton's First Law). (§383)

Regarding the second law, Hertz similarly shows that if a system with no connections accelerates (i.e. by being coupled with another system, hence having a force exerted on it), then that acceleration 'takes place in the direction of the force which acts on the system, and its magnitude is equal to the magnitude of the force, divided by the mass of the system' (§494). What this motion will look like will of course depend entirely on the force in question (i.e. how this connection-free system is coupled to some other system). However, in the limiting case where the system we are considering consists of just one material point, its acceleration can always be given by an ordinary three-dimensional vector. Hence Hertz takes the case of a single point—again as the limiting case of a system without connections—as corresponding to a statement of Newton's second law:

> The acceleration of a single material point takes place in the direction of the force acting on it, and its magnitude is equal to the magnitude of the force, divided by the mass of the point (Newton's second law). (§495)

[27]For Hertz, an isolated point is a rather special case: 'in reality, the material particle is simply an abstraction, whereas the material system is presented directly to us. All actual experience is obtained directly from systems; and it is only by processes of reasoning that we deduce conclusions as to possible experiences with single points' (Hertz, 1899, 31).

The case of the third law is more subtle: Hertz derives an action-reaction principle that has a close relationship with Newton's third law but is not equivalent to it (even for a single point). Earlier, we defined the vector Q_ρ as the force which the system B exerts on the system A. This is a vector quantity with regard to system A because it gives the components of the force along the coordinates q_ρ. If we wish to regard this force as a vector quantity with regard to system B, it's components along $\boldsymbol{q_\rho}$ can be denoted $\boldsymbol{Q'_\rho}$. Similarly, we can consider the "counterforce" that the system A exerts on the system B. The components of *this* force along the coordinates of system B are then denoted by $\boldsymbol{Q_\rho}$, and, as before, if we consider this force as a vector quantity with regard to system A, this is denoted by Q'_ρ. Hertz then shows (§467) that:

$$Q_\rho = -\boldsymbol{Q_\rho}$$ [28]

However, Hertz is careful to delineate this result from Newton's third law. The key difference lies in the fact that Newton's action-reaction principle, unlike Hertz's, is intended to apply to actions-at-a-distance:

> Newton's Law, as he intended it to be understood, contains our proposition completely; this is shown by the examples appended to his statement of the law. But Newton's Law contains more. At least it is usually applied to actions-at-a-distance, i.e. to forces between bodies which have no common coordinates. But our mechanics does not recognise such actions. (§469)

Hertz argues that where Newton's action-reaction principle deviates from his own is precisely where it is open to criticism. Firstly, unless the two bodies are point-masses, the line connecting them (along which the two forces are supposed to be equal and opposite) is open to ambiguity—for two extended bodies there is no immediate way to determine at what point and in what direction the forces in question act.[29] Secondly, as experimental results had been starting to show in Hertz's lifetime, the validity of the third law was called into question when applied to electromagnetic interactions between distant bodies.[30]

Most importantly for present purposes, Hertz's framework offers a thorough clarification of the relationship between Newtonian and Lagrangian forces. In line with

[28] Or, equivalently: $Q'_\rho = -\boldsymbol{Q'_\rho}$. Hence $Q_\rho = -Q'_\rho$, $\boldsymbol{Q_\rho} = -\boldsymbol{Q'_\rho}$.

[29] Karl Pearson expressed this concern as follows: 'the mutual action of two bodies is more complex than a reader just starting his study of mechanism would imagine, if he naturally interpreted mutual action as corresponding to mutual acceleration in some one line' (Pearson, 1957, p. 324).

[30] See §470.

his re-statement of Newton's laws of motion, Hertz identifies the vectorial notion of force with the force on a single point. This is the class of Hertzian forces which are represented by ordinary three-dimensional vectors. Thus Hertz recovers a picture of 'elementary mechanics': Newtonian (three-dimensional) forces acting on point-masses. In contrast, Hertz calls the more general (higher-dimensional) forces Lagrangian:

> As a rule, elementary mechanics means by forces only elementary forces. By way of distinction, the more general forms of forces hitherto considered by us are denoted as Lagrangian forces. Similarly we might denote the elementary forces as Galilean or Newtonian forces. (§476)

The fact that every system can be conceived as composed of material point in three-dimensional Euclidean space means that every Lagrangian force can be conceived as decomposable into elementary Newtonian forces (§479). Although in many cases such a decomposition is highly arbitrary, the key point is that, from the perspective provided by Hertz's framework, the relationship between general (Lagrangian) and elementary (Newtonian) forces is unproblematic.

In summary, Hertz's notion of force is introduced as a way to succinctly capture the effect that one system can have on another when the two are coupled together to form a combined system. In the general case, this is best represented by the high-dimensional Lagrangian forces that are typically tied to a system's degrees of freedom. Derivatively, we can recover the three-dimensional vectors in ordinary space that correspond to Netwonian forces, either on individual points making up a complex system, or on a single isolated point on its own.

Before concluding, let us return once more to Hertz's example of the swinging stone. Recall that we identified two potential obscurities in this case. The first was that appealing to centrifugal force—an inertial force experienced by an observer undergoing circular motion—seemed the wrong way to account for the reaction force from the stone on the hand. The second and deeper source of confusion, however, was linked to the introduction of inertial forces themselves. As Hertz wrote, 'In our laws of motion, force was a cause of motion, and was present *before* the motion. Can we, without confusing our ideas, suddenly begin to speak of forces which are a consequence of motion?' (Hertz, 1899, 6)

In the context of Hertz's formulation of mechanics this tension is resolved in the following way. As we have seen, according to Hertz's fundamental law, 'whenever two bodies belong to the same system, the motion of one is determined by that of the other' (Hertz, 1899, 28). The notion of force is then re-introduced because it is often convenient 'to divide the determination of the one motion by the other into two steps':

> We thus say that the motion of the first body determines a force, and that this force then determines the motion of the second body. In this way force can with equal justice be regarded as being always a cause of motion, and at the same time a consequence of motion. Strictly speaking, it is a middle term conceived only between two motions. (Hertz, 1899, 28)

From Hertz's perspective, we appeal to forces when we only have partial knowledge of the system at hand—when we do not know the details of the complete free system of which it is a part. Unsurprisingly, however, these are the kinds of systems that we encounter most often, and in fact they constitute many of the paradigm examples in the customary representation of mechanics.[31] Nevertheless, Hertz urges us to recognize that if we adopt the definition of force that his reformulation of mechanics makes available, the confusing aspects of forces will dissolve. In particular, we will be able to accept that, in a certain sense, forces can be regarded as causes of motion or as caused by motion, depending on our perspective on the particular problem at hand.

7 Conclusion

The goals of this paper have been two-fold. First, I hope to have unearthed the source of the 'logical obscurity' in the customary notion of force that troubled Hertz. Second, I hope to have shown how Hertz employed the apparatus of differential geometry to provide a unitary—hence unambiguous—notion of force. With regard to my first goal, I have argued that a subtle ambiguity in Newton's original laws of motion led to the emergence of two slightly different notions of force—Newtonian and Lagrangian—the relationship between which was obscured in the customary representation of mechanics. With regard to my second goal, I argued that to avoid these difficulties Hertz re-defined a force as the effect that one coupled sub-system has on the motion of another. With this in hand, Hertz was then able to recover logically perspicuous versions of Newton's three laws of motion and also able to show how Newtonian and Lagrangian forces can be related to one another in a clear

[31] As Hertz puts it, a system acted on by forces (or another kind of partial system) 'which is the remote and special case from the standpoint of our mechanics, is the commonest case in the problems which occur in daily life and in the arts' (Hertz, 1899, 39-40).

and unproblematic way. In passing, I noted that Hertz appeared to many of his readers to rely on an unusual and speculative ontology, particularly with regard to his introduction of hidden masses. But on this matter I argued that it was crucial to recognise that Hertz was only concerned to construct appropriate dynamical models of mechanical systems, and explicitly disavowed that these models would capture the ontological constitution of their targets. With this in view, we can appreciate that Hertz was deploying the mathematical apparatus of differential geometry to provide a uniform representation of all mechanical systems, so that all mechanical phenomena could be seen to fall under a single 'fundamental law'.

Recall that in offering a unitary notion of force, Hertz did not regard himself as having thereby answered the question: what is the *essence* of force? Rather, by disentangling the conflicting demands on the notion of force that arose in the customary formulation of mechanics, Hertz believed that confused questions as to its essence would no longer arise (that 'our minds, no longer vexed, would cease to ask illegitimate questions'). One final hope of this paper is to have shed some light on what this avoidance of confused questions amounted to.

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