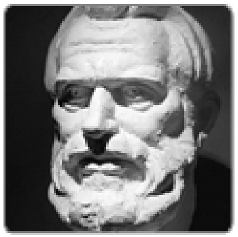

Epistemology of Geometry

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Introduction

From Euclid to Einstein, geometry has continually shown itself to be a remarkably rich area of philosophical inquiry. Major geometrical traditions find their roots in the *Āryabhatīya of Āryabhata* from the classical age of Indian mathematics, the *Nine Chapters on the Mathematical Art* from Han dynasty China, and in Euclid's *Elements* of ancient Greece, conveyed via the Arabic textual tradition to medieval scholars. Ancient reasoning methods—characteristically relying on pictures or diagrams—have found a renaissance in the scholarship of the last few decades and are now seen as far more rigorous than has often been supposed. Much contemporary geometry traces back to the development of algebraic methods (spurred in the Western tradition through philosophers such as Descartes and Leibniz), and geometry today overlaps with many other areas of mathematics (including set theory, category theory, and homotopy type theory). In the interim, a number of traditions have risen to prominence. Kant regarded geometry as the paradigm example of the synthetic a priori, and Hilbert's work on geometry set a new standard for the axiomatization of a mathematical theory. Having always been admired as a paradigm of science, there is a long-standing tradition of seeking out "geometrical formulations" of physical theories (including classical mechanics, electromagnetism, and quantum physics). The emergence of a plurality of non-Euclidean geometries—sending shock waves through mathematics, physics and philosophy—led in particular to the 19th-century "problem of space," which then fed into Einstein's theories of special and general relativity, radically transforming the physical application of geometrical notions. This article is organized into three major areas of the epistemology of geometry. The first, Geometrical Reasoning, concerns the epistemology of geometrical practice itself. The second, Geometry and Philosophy, concerns the various pathways between geometry and philosophical theorizing more generally, including the historical engagement between the two. And the third, Geometry and Physics, concerns the intimate connections between geometry and physics, especially (of course) the physics of space. The bibliography begins with a selection of general overviews and textbooks that can serve as entry points into more specific areas.

General Overviews

These overviews cover some of the most important periods in the history of geometry. Mueller 2012 discusses the context of Euclid's *Elements*. Sarasvati Amma 1999 aims to correct the historical error that Indian mathematicians of the premodern period were brilliant algebraists but did not make major strides in geometry. Klein 1893 and Torretti 1978 both discuss the major advances in geometry during the 19th century, and Friedman 1983 surveys a novel approach to geometry that emerged in the aftermath of relativity theory.

Friedman, Michael. *Foundations of Spacetime Theories: Relativistic Physics and Philosophy of Science*. Princeton, NJ: Princeton University Press, 1983.

A rigorous but accessible account of the overarching conflict between conventionalism and realism concerning spacetime geometry. Friedman applies a four-dimensional approach (embedded in the formal apparatus of differential geometry) to classical physics, special relativity, and general relativity, providing a gateway into the contemporary philosophy of spacetime physics.

Klein, Felix. "A Comparative Review of Recent Researches in Geometry." *Bulletin of the American Mathematical Society* 2.10 (July 1893): 215–249.

A survey of the various branches of geometry in the second half of the 19th century by one of the masters of the period. Topics covered include group theory, projective geometry, line geometry, and *analysis situs*. Klein's text offers insights into the evolving subject matter of geometry itself and the relationship between geometry and other branches of mathematics.

Mueller, Ian. "Greek Mathematics to the Time of Euclid." In *A Companion to Ancient Philosophy*. Edited by Mary Louise Gill and Pierre Pellegrin, 686–718. Oxford: Blackwell, 2012.

Mueller's paper establishes the sources of Euclid's *Elements* in the Greek tradition, including in the traditions of Plato, Aristotle, and the Pythagoreans. The first part of the paper focuses on plane geometry, including the relation between geometry and algebra; the second part deals with the history of Greek arithmetic; and the final part delves into lesser known figures and traditions in Greek geometry.

Sarasvati Amma, T. A. *Geometry in Ancient and Medieval India*. Delhi: Indological Publications, 1999.

A classic text on ancient and medieval Indian geometry. Surveys the Sanskrit and Prakrit traditions, including an early focus on the canonical texts of the Sulbasutras in the Vedic literature. Deals in depth with the intimate connections between geometry and astronomy in those traditions, in particular in the work of well-known Indian geometers such as Aryabhata, Sripati, and Bhaskara. The historical account extends to the early 17th century.

Torretti, Roberto. *Philosophy of Geometry from Riemann to Poincaré*. Dordrecht, The Netherlands: Springer, 1978.

A landmark review and analysis of the history of the philosophy of geometry, focused on the major developments of the 19th century. Torretti provides a brief background on ancient geometry; a lucid discussion of metric, projective, and axiomatic geometry; and an analysis of empiricist and conventionalist interpretations of geometry.

Textbooks

Gray 2011 covers the history of geometry in the 19th century, a period of particularly intense activity in this area. The other two texts in this section concern the interaction between geometry and physics: Carroll 2003 is an introductory textbook for general relativity, and Arnold 2010 is a relatively advanced textbook for classical mechanics, demonstrating a number of the advantages of a "geometrical formulation" of a theory.

Arnold, V. I. *Mathematical Methods of Classical Mechanics*. 2d ed. Translated by K. Vogtmann and A. Weinstein. New York: Springer Science+Business Media, 2010.

A classic textbook for classical mechanics that emphasizes a geometrical formulation of the theory, particularly in the setting of a *symplectic manifold*.

Carroll, Sean M. *Spacetime and Geometry: An Introduction to General Relativity*. San Francisco: Pearson, 2003.

A lucid introductory textbook for general relativity, covering the mathematics and physics that feeds into the vast philosophical literature on this topic.

Gray, Jeremy. *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*. London: Springer-Verlag London, 2011.

The leading textbook on the history of geometry in the 19th century. Gray discusses the relationships that geometry has to other branches of mathematics, to physics, and to philosophy, and includes a final chapter on the topic of "geometrical truth."

Geometrical Reasoning

There is a long-standing tradition of taking geometrical reasoning as a paradigm for reasoning in philosophy and in logic, and vice versa. Inferences drawn in geometry might depend on axioms as rules of reasoning, postulates, construction procedures, or elucidations of terms such as “point,” “line,” and “plane.” The Euclidean tradition in the West (and extending outward), the tradition of the *Nine Chapters* in China, and the Vedic and Kerala traditions in medieval India, among others, founded lineages of geometrical methods for reasoning that had a profound influence on methods of proof and inference quite generally. Perhaps most prominently, the “axiomatic method” in geometry has been regarded as a model of reasoning in both philosophy and science, and the linkages between geometrical and epistemological methods of proof and inference linger to the present day.

The Euclidean Tradition

As Mueller 2012 (cited under General Overviews) makes clear, the geometrical tradition founded on Euclid 1908 (originally published *c*. 300 BCE) and drawing on work from Aristotle and Plato was remarkably unified up until the development of non-Euclidean geometries precipitated by works such as Saccheri 1920 (originally published in 1733). Netz 1999 shows how Greek methods of deduction and proof were shaped by linguistic and diagrammatic practices in geometry; de Young 1984 and Mehdi and van Brummelen 2017 analyze responses to and commentaries on Euclid in the Arabic world; while Layne and Butorac 2017, Harari 2006, and Cantù 2010 evaluate the tradition of Euclid reception going through Proclus. Beyond this initial reception tradition, Laywine 1998 and Dunlop 2009 show that philosophers, prominently including Immanuel Kant, have regarded Euclid 1908 as a paradigm for reasoning in general (see Geometry and Philosophy).

Cantù, Paola. “Aristotle’s Prohibition Rule on Kind-Crossing and the Definition of Mathematics as a Science of Quantities.” *Synthese* 174.2 (2010): 225–235.

An article interpreting Proclus’s commentary to Euclid’s first book of *Elements*, among other texts, in the Aristotelian tradition.

De Young, Gregg. “The Arabic Textual Traditions of Euclid’s *Elements*.” *Historia Mathematica* 11.2 (1984): 147–160.

Uses archival work to suggest revisions to the then-standard reading of the translation and reception of Euclid’s *Elements* in the medieval Arab world.

Dunlop, Katherine. “Why Euclid’s Geometry Brooked No Doubt: J. H. Lambert on Certainty and the Existence of Models.” *Synthese* 167.1 (2009): 33–65.

Explains why J. H. Lambert (b. 1728–d. 1777), an important figure in the early history of non-Euclidean geometry, did not think that his results threatened the “certainty” of the Euclidean method. On Dunlop’s reading, Lambert argues that Euclid establishes certainty and removes doubt by the use of postulates and models.

Euclid. *Elements*. Translated by Sir Thomas Little Heath. Hathi Trust Digital Library. Cambridge, UK: Cambridge University Press, 1908.

The contemporary notion of an axiomatic treatment of a subject traces its roots back to the paradigm provided by Euclid’s work: a systematic deductive treatment of plane geometry, solid geometry, and elementary number theory. Euclid’s *Elements* is the single most influential text in the history of geometry, as well as a hugely influential text in the history of logic and science.

Harari, Orna. “Methexis and Geometrical Reasoning in Proclus’ Commentary on Euclid’s *Elements*.” In *Oxford Studies in Ancient Philosophy*, Vol. 30. Edited by David Sedley, 361–390. Oxford: Oxford University Press, 2006.

Reads Proclus's commentary on Euclid's elements as separating geometrical knowledge from geometrical proof: "Geometrical knowledge is about discursive objects while geometrical proofs deal with imagined objects" (p. 362). This leads to two questions, analyzed in the paper: (1) how, then, do we deal with the "particularity and constructibility" of geometrical objects, and (2) does Proclus in fact provide "an adequate analysis of Euclid's geometrical reasoning" (p. 363)?

Layne, Danielle A., and David D. Butorac, eds. *Proclus and His Legacy*. Berlin: De Gruyter, 2017.

A volume focusing mainly on the reception and transmission of Proclus's work, in contexts ranging from late Neoplatonic, Alexandrine and Byzantine philosophy to Jewish, Arabic, and Islamic thought. A valuable resource for situating Proclus's work within these scholarly traditions.

Laywine, Alison. "Problems and Postulates: Kant on Reason and Understanding." *Journal of the History of Philosophy* 36.2 (1998): 279–309.

Analyzes Kant's use of Euclidean methods as models of reasoning in philosophy, as well as in geometry and mathematics generally.

Mehdi, Aminrazavi, and Glen Van Brummelen. "Umar Khayyam." In *The Stanford Encyclopedia of Philosophy*. Edited by Edward N. Zalta. Stanford, CA: Stanford University Press, 2017.

Khayyam's discussion of Euclid's infamous parallel postulate can now be recognized as the first substantial work in non-Euclidean geometry. An important aspect of Khayyam's work is the development of the so-called Khayyam-Saccheri quadrilateral, which can be used to distinguish between Euclidean, hyperbolic, and elliptic geometries.

Netz, Reviel. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge, UK: Cambridge University Press, 1999.

Netz's study is built on a careful investigation of Hellenistic geometry, revealing the deductive strategies of the Greeks via their linguistic practices and diagrammatic reasoning methods. Netz shows that the way Greek mathematicians labeled and spoke about diagrams can illuminate their methods of inference and proof. In the concluding chapters, Netz traces how the notion of the generality of a proof was built up through the use of (fallible) formulae and increasingly general language. Netz's work is influenced by that of Ian Mueller, and profitable connections can be drawn between his work and that of Karine Chemla on proof and generality in the Asian tradition

Saccheri, Girolamo. *Euclides Vindicatus*. Translated by George Bruce Halsted. Chicago: Open Court, 1920.

Saccheri's work precipitated the full development of non-Euclidean geometries in the 19th century. Issues raised in Saccheri's discussion include the standards for an acceptable mathematical definition and what it could mean to "prove" an axiom. Although leading to a fallacious proof of Euclid's parallel postulate, many of Saccheri's results are now recognized as theorems in hyperbolic geometry.

Pre-Modern Chinese Geometry

The Nine Chapters on the Mathematical Art is a work from the Han dynasty, as fundamental to Chinese geometry as the *Elements* was elsewhere. Kangshen, et al. 2000 is a contemporary translation and edition including the influential 3rd century commentary by Liu Hui, which is a major topic of Chemla 1997. Su-lyn Lim, et al. 2017 and Chemla 2018 provide up to date scholarship on the continuations of the Chinese tradition in mathematics, and in geometrical reasoning and proof in particular, in the 7th century (Su-lyn Lim, et al. 2017) and the 11th to the 13th centuries (Chemla 2018).

Chemla, Karine. "What Is At Stake in Mathematical Proofs From Third-Century China?" *Science in Context* 10.2 (1997): 227–251.

Investigates the 3rd-century commentary on the Nine Chapters by Liu Hui, including the relations between philosophy and mathematics in the tradition of commentary. Presents a platform for an "international history of mathematical proof" (p. 227).

Chemla, Karine. "The Proof Is in the Diagram: Liu Yi and the Graphical Writing of Algebraic Equations in Eleventh-Century China." *Science in Context* 42.2–3 (2018): 60–77.

Chemla focuses on the geometrical diagrams used in algebraic proofs in 11th- to 13th-century China, arguing that "these diagrams constitute the proof of the correctness of the algorithms" used to obtain results (p. 60).

Kangshen, Shen, John N. Crossley, and Anthony W.-C. Lun, eds. and trans. *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford: Oxford University Press, 2000.

The anonymous treatise *The Nine Chapters* is as fundamental to the history and practice of Chinese mathematics as Euclid's *Elements* is to Western mathematics. It centers on 246 problems. This edition presents the text alongside later commentaries, especially the famous third century commentary by Liu Hui (see Chemla 1997 and Chemla 2018), which provide methods for solving the problems and explanations of those methods. This is the first full English translation of the work and is accompanied by scholarly commentary and editorial apparatus.

Su-lyn Lim, Tina, and Donald B. Wagner. *The Continuation of Ancient Mathematics: Wang Xiaotong's "Jigu suanjing" Algebra and Geometry in 7th-Century China*. Copenhagen, Denmark: NIAS Press, 2017.

While knowledge of Euclid dwindled in Europe following the collapse of the Roman Empire, uses of geometry in China continued to evolve, including new uses of geometric constructions in algebraic contexts. Lim and Wagner's in-depth study of Wang Xiaotong's canonical text explores a stage in this development from the 1st to the 14th century CE.

Pre-Modern Indian Geometry

Indian approaches to geometry in the ancient and medieval periods (including in the Vedic tradition and the Madhava or Kerala school) worked from the engagement of Indian mathematicians with astronomy, and astronomers with mathematics. This is seen in works such as the *Āryabhatīya of Āryabhata*. Approximations of the sine, the secant method of iterative approximation, spherical trigonometry, and the like were developed in this context. Plofker 2009 is a well-researched history of mathematics in India that should be a landmark text on the subject. Dutta 1932 is a comprehensive history of the Sulbasutras, the Hindu texts that founded an influential tradition in premodern Indian geometry.

Clark, Walter Eugene. *The Āryabhatīya of Āryabhata. An Ancient Indian Work on Mathematics and Astronomy. Translated with Notes*. Chicago: University of Chicago Press, 1930.

A landmark work in Indian astronomy and geometry. The later geometer Bhaskara contributed an influential commentary on this work (see Sarasvati Amma 1999, cited under General Overviews). A foundation of the Kerala school of astronomy and an influence on the connection between Indian and Islamic geometry (see Plofker 2009).

Dutta, Bibhutibhusan. *The Science of the Sulba: A Study in Early Hindu Geometry*. Calcutta: University of Calcutta, 1932.

A comprehensive study of the Sulbasutras of the Vedic tradition in early Indian geometry and astronomy, a tradition from which many of the greatest achievements of premodern Indian mathematics emerged.

Plofker, Kim. *Mathematics in India*. Princeton, NJ: Princeton University Press, 2009.

An engagement with Indian mathematics from the early Sanskrit tradition and the Sulbasutras in the Vedic texts, to the Madhava school in Kerala from the 14th to the 17th centuries, to engagement with Islamic mathematics, up to the development of the modern period in Indian mathematics. The book includes a deep discussion of the relationship between text and commentary in Indian geometry, astronomy, and mathematics.

Euclidean Methods in Contemporary Philosophy of Mathematics

Diagrammatic methods are enjoying a renaissance in contemporary philosophy of mathematics, and Euclidean methods accordingly are gaining steam. Manders 2008 is an influential investigation into Euclidean reasoning using diagrams. Avigad, et al. 2009 continues the analysis of Euclidean methods begun in Manders 2008, bringing the analysis forward to formal axiomatic systems, and Mumma 2010 investigates the epistemic and methodological conditions for Euclidean diagrammatic proof.

Avigad, Jeremy, Edward Dean, and John Mumma. "A Formal System for Euclid's *Elements*." *Review of Symbolic Logic* 2.4 (2009): 700–768.

Drawing on Kenneth Manders's work, the authors offer a detailed analysis of Euclid's inference methods, arguing that diagrammatic reasoning in the *Elements* is not only controlled and systematic but can be reflected in a contemporary formal axiomatic system that is sound and complete.

Manders, Kenneth. "The Euclidean Diagram." In *The Philosophy of Mathematical Practice*. Edited by Paolo Mancosu, 80–133. Oxford: Oxford University Press, 2008.

Circulating for a significant time before it was published, Mander's highly influential paper led to renewed interest in the nature, rigor, and limitations of Euclidean diagrammatic reasoning.

Mumma, John. "Proof, Pictures, and Euclid." *Synthese* 175.2 (July 2010): 255–287.

From the contemporary perspective, Euclid's diagrammatic reasoning—or any reasoning that relies essentially on *pictures*—can seem to fall far short of the standards of mathematical rigor. After describing and analyzing this received view, Mumma outlines a set of necessary conditions for a genuine "picture proof" and provides a formal apparatus to show how a particular diagram can warrant general conclusions.

Nineteenth and Twentieth Century Epistemology and Geometry

The 19th century saw a remarkable number of major developments pertaining to the epistemology of geometry. These included the rise of non-Euclidean geometry, an increase in the use of geometry to model physical phenomena (such as in geometrical optics), an ever-increasing connection between (and mutual enrichment of) algebraic and geometrical methods, the increasing use of axiomatized formal systems, and the development of group theory and projective methods in geometrical reasoning. All these trends continued to develop in the 20th century, and the truly central role of geometry in epistemology and physics was cemented by the inauguration of the theories of special and general relativity. Russell 1891 provides a well-known, if flawed, history of geometry, and a much more well-founded account of the interrelation between geometry and cognate fields. A more recent overview is provided by Biagioli 2016, focusing on the expanding connections between algebra, group theory, and geometry, not to mention the relationships with both physics and epistemology. Especially influential in the 19th and 20th centuries was the rise of axiomatization programs. Geometers who wished to base their systems on empirical (or, as David Hilbert put it, concrete) "intuition" or observation could appeal to axioms as rules for inference, to show how concrete observations can be built into proofs using rules and construction procedures. An early, rigorously empirical system by Pasch is explored in Schlimm 2010, whereas Zach 2016 presents the foundations of Hilbert's program for classical mathematics. Blanchette 2018 addresses the dispute between Hilbert and Frege over the foundations of geometry, and Corry 2004 is a comprehensive history of Hilbert's axiomatization of physics, covering the genesis of Hilbert's formulation of his equations for general relativity, among other results.

Biagioli, Francesca. *Space, Number, and Geometry from Helmholtz to Cassirer*. Dordrecht, The Netherlands: Springer, 2016.

Analysis of the interplay between geometrical method, algebra, and physics during the 19th and early 20th centuries, focusing on the careers of Hermann von Helmholtz and Ernst Cassirer but drawing in work on figures including Karl Georg Christian von Staudt, Jean-Victor Poncelet, Sophus Lie, Felix Klein, Richard Dedekind, and Henri Poincaré.

Blanchette, Patricia. "The Frege-Hilbert Controversy." In *The Stanford Encyclopedia of Philosophy*. Edited by Edward N. Zalta. Stanford, CA: Stanford University, 2018.

A survey and analysis of work on the dispute between Gottlob Frege and David Hilbert over the axiomatization of geometry, a dispute that turned on epistemological questions at least in part.

Corry, Leo. *David Hilbert and the Axiomatization of Physics (1898–1918). From Grundlagen der Geometrie to Grundlagen der Physik*. Dordrecht, The Netherlands: Springer, 2004.

A thorough and meticulously researched presentation of David Hilbert's program for the axiomatization of physics in the early 20th century.

Russell, Bertrand. *An Essay on the Foundations of Geometry* Project Gutenberg. London: C. J. Clay and Sons, 1891.

Based on his dissertation for a fellowship at Trinity College, Russell discusses the history and philosophy of geometry from Kant up until the end of the 19th century, the differences between projective and metric geometry, and how geometry lies at the intersection between mathematics, physics, psychology, and philosophy.

Schlimm, Dirk. "Pasch's Philosophy of Mathematics." *Review of Symbolic Logic* 3.1 (2010): 93–118.

A clear exposition of the philosophy of mathematics of Moritz Pasch, who provided "the first rigorous axiomatization of projective geometry" (p. 93) in 1882. Pasch's epistemology combined deductivism and empiricism and was a landmark in the development of the field.

Zach, Richard. "Hilbert's Program. In *The Stanford Encyclopedia of Philosophy*. Edited by Edward N. Zalta. Stanford, CA: Stanford University Press, 2016.

An encyclopedia entry covering the axiomatic foundation for classical mathematics proposed by David Hilbert in the early 1920s. Provides an overview of Hilbert's program, from its initial formulation by Hilbert himself, to contributions in the 1920s and 1930s by others including Paul Bernays, Wilhelm Ackermann, John von Neumann, Jacques Herbrand, and even Kurt Gödel, to its influence on the development of proof theory.

Geometry and Philosophy

The relationship between geometry and philosophy is by no means exhausted by detailing the philosophy *of* geometry. Historically, geometry and philosophy have developed in tandem, with each serving the other as a source of methods, cases, problems, and models, and with each providing stimuli and support for major results and traditions (here, the Cartesian tradition, and the development of the calculus, are key). This section begins with a historical survey that provides some high points of the historical co-development of geometry and philosophy. The second section continues the discussion of Hilbert's programs, but now from the perspective of contemporary work aiming to revive Hilbert's research. The final section looks to the future of the intersection between philosophy, geometry, and category and set theory.

Historical Development of Geometry and Philosophy

As Menn 2002 details, geometrical methods have often been taken as models for philosophical reasoning. Ighbariah and Wagner 2018 explains that philosophical or theoretical commitments by mathematicians and physicists can influence their interpretations of and even revisions to geometrical and mathematical methods and provides an analysis of Ibn al-Haytham's revisions to Euclidean methods to be more in line with his empirical, experimentally based commitments. De Risi 2007 analyzes Leibniz's analysis situs, and Grosholz 2016 evaluates Leibniz's transcendental curves, both drawing broader conclusions about his methods and philosophy of space. Heis 2015 considers Kant's use of "real definitions" in geometry as part of his philosophical account of geometry as synthetic a priori. Tappenden 2006 finds a "revolution" in geometry in the 19th century, identifying a Riemannian background to Frege's work. (Note here an interesting connection to Norton 1999.) Schiemer 2018 presents the "structural turn" in modern geometry as a key influence on Ernst Cassirer's neo-Kantian epistemology. Richardson 2003 takes a broader group (Lewis, Becker, and Carnap) as the central subject, arguing that they used methods and results from geometry to inform philosophical strategies.

De Risi, Vincenzo. *Geometry and Monadology: Leibniz's Analysis Situs and Philosophy of Space*. Dordrecht, The Netherlands: Springer, 2007.

A far-reaching history of Leibniz's views on geometry in historical context, including the relationship between Leibniz's geometry and his philosophy.

Grosholz, Emily. "Leibnizian Analysis, Canonical Objects, and Generalization." In *The Oxford Handbook of Generality in Mathematics and the Sciences*. Edited by Karine Chemla, Renaud Chorlay, and David Rabouin, 329–344. Oxford: Oxford University Press, 2016.

Focuses on Leibniz's analysis of transcendental curves. Argues on that basis that Leibniz was aiming, in his mathematical work, at a general method of analysis and explanation, rather than (exclusively) aiming at abstraction and formal proof.

Heis, Jeremy. "Kant (vs. Leibniz, Wolff and Lambert) on Real Definitions in Geometry." *Canadian Journal of Philosophy* 22.5–6 (2015): 605–630.

Heis examines Kant's conception of a "real" geometrical definition, especially the definitions of circles and parallel lines, and tracks the subtle (but philosophically significant) differences between Kant and his contemporaries on this matter. This paper also helps contextualize 19th- and 20th-century criticisms of Kant's conception of geometry as a body of synthetic a priori knowledge.

Ighbariah, Ahmad, and Roy Wagner. "Ibn al-Haytham's Revision of the Euclidean Foundations of Mathematics." *HOPOS: The Journal of the International Society for the History of Philosophy of Science* 8.1 (2018): 62–86.

A study of how the 10th-century mathematician, astronomer, and physicist Ibn al-Haytham's treatment of the Euclidean common notions or axioms was part of a larger revision of the Euclidean program to be more in line with al-Haytham's own empirical, experimental approach.

Menn, Stephen. "Plato and the Method of Analysis." *Phronesis* 47.3 (2002): 193–223.

Explores how far in the past philosophers took geometry (and the method of analysis in particular) as a model for philosophical reasoning, with a special focus on Plato's Meno.

Richardson, Alan. "The Geometry of Knowledge: Lewis, Becker, Carnap and the Formalization of Philosophy in the 1920s." *Studies in History and Philosophy of Science Part A* 34.1 (2003): 165–182.

Argues that science is not merely a topic for philosophical reflection, but also "provides resources" for philosophical methods and approaches. Section 2 focuses on cooperation between Rudolf Carnap and Oskar Becker, both of whom were working on pure and applied geometry in the 1920s, and explores how Carnap and Becker "used the lessons of geometry to motivate new understandings of the issues in philosophy of the exact sciences and . . . epistemology" (p. 175).

Schiemer, Georg. “Cassirer and the Structural Turn in Modern Geometry.” *Journal of the History of Analytical Philosophy* 6.3 (2018): 182–212.

A deeply thoughtful analysis of the relation between modern geometry, especially fundamental results in projective geometry, and the neo-Kantian epistemology of Ernst Cassirer.

Tappenden, James. “The Riemannian Background to Frege’s Philosophy.” In *The Architecture of Modern Mathematics: Essays in History and Philosophy*. Edited by J. Ferreirós and J. J. Gray, 97–132. Oxford: Oxford University Press, 2006.

Argues that there was a mathematical revolution in the 19th century, which philosophers and historians have failed to recognize as such. A well-researched history of the mathematical background to Frege’s philosophy of mathematics, including an argument that Frege was influenced by the Riemannian tradition.

Contemporary Work on Hilbert’s Program

Kurt Gödel’s incompleteness theorems were a major blow to Hilbert’s program as formulated initially. Nonetheless Hilbert’s program has defenders in the contemporary landscape of philosophy of mathematics, some of whom deny the impact of Gödel’s theorems on the program itself. The significance of Hilbert’s programs, of meta-mathematics, and of axiomatization to the epistemology of geometry is hard to overstate, since the results were used almost immediately to demonstrate consistency of the axiomatic systems of geometry relative to one another, to show the independence or dependence of geometrical systems and results on one another, and to show how proof theory can be built in geometry. Zach 2006 surveys Hilbert’s programs then and now. Detlefsen 2013 reads Hilbert as a mathematical instrumentalist and argues that this reading allows for most of Hilbert’s program to be salvaged. Sieg 2013 explains the view that Hilbert articulated. Franks 2009 reads Hilbert as a naturalist who aimed to prove the autonomy of mathematics from philosophy, and, especially, from the demand for philosophical foundations of the kind Frege and Brouwer required. Patton 2014 argues that Hilbert can demonstrate the objectivity of mathematical statements even if we attribute to him what is usually called “formalism,” namely, the idea that mathematical terms denote arbitrary, meaningless signs.

Detlefsen, Michael. *Hilbert’s Program: An Essay on Mathematical Instrumentalism*. New York: Springer, 2013.

A sustained argument for the claim that the aims of Hilbert’s program include using mathematical signs and methods as instruments to demonstrate the objectivity of mathematical statements, and a clear and deep presentation of what finitist mathematics would need to look like to support the aims of Hilbert’s program.

Franks, Curtis. *The Autonomy of Mathematical Knowledge: Hilbert’s Program Revisited*. Cambridge, UK: Cambridge University Press, 2009.

A reading of Hilbert according to which he was more interested in pursuing his goals in meta-mathematics, in proof theory, and in consistency results, than he was in providing philosophical foundations for mathematics (as Frege and Brouwer demanded). In fact, Franks goes further to argue that Hilbert aimed at the autonomy of mathematics (and geometry) from philosophy.

Patton, Lydia. “Hilbert’s Objectivity.” *Historia Mathematica* 41.2 (2014): 188–203.

Hilbert’s controversial use of “arbitrary” signs in his axiomatic systems (see Blanchette 2018 cited under Nineteenth and Twentieth Century Epistemology and Geometry) is shown to be backed up by a method for using signs to prove statements with objective content, statements that can be proven true or false. That method was used earlier in the 19th century by physicists including Hermann von Helmholtz. Hilbert’s “formalism” and his use of “arbitrary” signs is shown not to be a mistake but rather a significant move forward in a tradition deeply rooted in 19th-century physics and mathematics.

Sieg, Wilfried. *Hilbert's Programs and Beyond*. Oxford: Oxford University Press, 2013.

Sieg traces the roots of Hilbert's programs in his engagement with the work of Dedekind and 19th-century mathematics generally. He explains how finitist proof theory and meta-mathematics were developed and the significance of Gödel's results for Hilbert's research program. Sieg concludes with two parts delving into the prospects for proof theory, reverse mathematics, theory reduction, and allied developments of Hilbert's research methods for future mathematics and philosophy of mathematics.

Zach, Richard. "Hilbert's Program Then and Now." In *Philosophy of Logic*. Handbook of the Philosophy of Science, Vol. 5. Edited by Dale Jacquette, 411–447. Amsterdam: Elsevier, 2006.

Provides a historical overview of the development of Hilbert's program, with a particular view to presenting the motivation of the program: first, to respond to criticisms from Brouwer and others that mathematics requires principles such as induction that cannot be proven, and secondly, to show through meta-mathematical proofs that no contradiction can be derived within mathematics. Zach concludes with a discussion of the prospects for Hilbert's program in contemporary work.

Geometry, Set Theory, and Category Theory

Set theory and category theory have developed in connection with geometrical reasoning. It is not only that geometry is used as an example of a kind of reasoning, or that geometrical objects are the constituents of sets. Rather, geometrical reasoning in practice provides models of proof structures, and group actions in geometry, for instance, provide distinct fields of research for category and set theory. McLarty 1996 is a textbook for learning category theory and how to approach problems in the field. Lawvere 2005 sets out a landmark presentation of the foundations of category theory, while Schneps and Lochal 2013 is exactly that: a road map for a generalization of Galois theory, via deep analysis of the geometry of real surfaces and the algebraic geometry of algebraic curves. Landry 2017, and in particular Corfield 2017, analyze the applications of category theory, homotopy type theory, and related programs in geometry and in mathematics generally, and the significance of these for philosophy.

Corfield, David. "Reviving the Philosophy of Geometry." In *Categories for the Working Philosopher*. Edited by Elaine Landry, 18–35. New York: Oxford University Press, 2017.

Distinguishing between reflection on the history of philosophy of geometry from the activity of philosophy of geometry itself, Corfield calls for a revival of the latter equipped for contemporary mathematical developments, especially homotopy type theory. Provides a survey of the work that stands to be pursued in this contemporary setting and argues for the possibility of a revised conception of geometry itself.

Landry, Elaine, ed. *Categories for the Working Philosopher*. Oxford: Oxford University Press, 2017.

A book covering the most basic and significant applications of category theory, beginning from the position that "what makes categories interesting and significant is their specific use for specific purposes" (from the book description). Important segments of the book (Corfield 2017, in particular) cover the application of categories to geometry.

Lawvere, F. William. "An Elementary Theory of the Category of Sets (Long Version) with Commentary." *Proceedings of the National Academy of Science of the U.S.A 52*, 1506–1511, 2005.

A classic work in the foundations of the category theory of sets, with commentary. Republished in *Reprints in Theory and Applications of Categories* 11: 1–35. An expanded version, with commentary by Colin McLarty and the author, of "An Elementary Theory of the Category of Sets."

McLarty, Colin. *Elementary Categories, Elementary Toposes*. Oxford: Clarendon Press, 1996.

A textbook introducing the theory of categories, the theory of functors, and topos theory. Works from a category theory basis throughout without assuming set theory.

Schneps, Leila, and Pierre Lochal, eds. *Geometric Galois Actions: Around Grothendieck's Esquisse d'un Programme*. Vol. 1. Cambridge, UK: Cambridge University Press, 2013.

The mathematician Alexander Grothendieck provided a "sketch" of a program for category theory, which is now receiving significant attention. Schneps and Lochal present a reprint of Grothendieck's sketch in the original French and in English translation, accompanied by a substantial collection of essays on the topic, focusing on geometric group actions. Grothendieck begins from algebraic geometry, focusing on the geometry of real surfaces. He arrives at profound generalizations of the Galois group. Requires mathematical training to understand fully but can be read for the games as well.

Geometry and Physics

Einstein once called geometry "the oldest branch of physics", and many of the oldest (and more recent) topics of interest in the epistemology of geometry stem from the intimate connection between the two fields. The following section is divided into four sub-sections: The Foundations of Physical Geometry, Space and Spacetime, Geometrical Formulations of Physical Theories, and Geometry and Symmetry.

The Foundations of Physical Geometry

Entries in this section include canonical historical discussions that followed the development of non-Euclidean geometries, including Helmholtz 1876, Poincaré 1898, and—perhaps most famous of all—Riemann's *Habilitationsvortrag* inaugural lecture in the form of Riemann 2016. Friedman 2002 surveys the transformation brought about by relativity theory, including a critique of the views of the logical empiricists. Broader surveys of the history of physical geometry are offered in article form by Stein 1977 and in book form by DiSalle 2006. More detailed analyses, concerning Gauss and Einstein in particular, are offered by Gray 2006 and Norton 1999 respectively.

DiSalle, Robert. *Understanding Spacetime: The Philosophical Development of Physics from Newton to Einstein*. Cambridge, UK: Cambridge University Press, 2006.

An elegant historical survey of the special nature of spacetime structure in physical theorizing, spanning both classical and relativistic physics. DiSalle draws particular attention to the way in which spacetime plays the role of a "framework" structure, within which other physical theories can be formulated, thereby providing an important contrast to Brown 2005 (cited under Space and Spacetime).

Friedman, Michael. "Physics, Philosophy and the Foundations of Geometry". *Diálogos* 79 (2002): 121–143.

Beginning with a critique of the logical empiricist distinction between pure and applied geometry. Friedman situates Einstein's development of special and general relativity against the backdrop of the debate between Poincaré's conventionalism and Helmholtz's empiricism, a debate that was itself intimately linked to the group-theoretic approaches to geometry of Lie and Klein. The outcome, Friedman argues, is a clearer insight into the seismic conceptual shift affected by general relativity.

Gray, Jeremy. "Gauss and Non-Euclidean Geometry." In *Non-Euclidean Geometries: Mathematics and Its Applications*. Edited by András Prékopa and Emil Molnár, 61–80. Vol. 581. Boston: Springer, 2006.

After reviewing the limited evidence concerning whether Gauss should be regarded as one of the co-discoverers of non-Euclidean geometry, Gray concludes that, at best, Gauss's instincts in this case were more that of a scientist than a mathematician. Along the way, Gray also analyses what it is to "discover a new geometry of space."

Helmholtz, Herman von. "On the Origin and Significance of Geometrical Axioms" *Mind* 1.3 (July 1876): 301–321.

Advancing a neo-Kantian treatment of the relationship between space and geometry, Helmholtz argues forcefully that the classical (hyperbolic and elliptic) non-Euclidean geometries are candidate descriptions of spatial experience and that the behavior of rigid bodies and light rays underpins the possibility of physical geometry. A masterful and readable discussion of the operationalization of geometrical notions.

Norton, John D. "Geometries in Collision: Einstein, Klein and Riemann." In *The Symbolic Universe: Geometry and Physics 1890–1930*. Edited by Jeremy J. Gray, 128–144. Oxford: Oxford University Press, 1999.

Norton argues that although special relativity is naturally associated with Klein's group-theoretic approach to geometry, the shift to general relativity relies essentially on Riemann's metric approach. Norton then argues that Einstein's understanding of the significance of the general covariance of general relativity stems from his ill-advised efforts to retain Klein's group-theoretic understanding of what is physically or geometrically "real."

Poincaré, Henri. "On the Foundations of Geometry." *The Monist* 9.1 (October 1898): 1–43.

Poincaré presents his own distinctive understanding of the relationships between geometrical reasoning, physical space, and visual and kinaesthetic sensations, concluding with a canonical statement of geometrical conventionalism.

Riemann, Bernhard. *On the Hypotheses Which Lie at the Bases of Geometry*. Edited by Jürgen Jost. Cham, Switzerland: Springer International Publishing, 2016.

Though relatively short and untechnical, Riemann's inaugural lecture ("*Habilitationsvortrag*") is one of the most important works in the history of geometry. Besides laying the foundation for contemporary differential geometry, it offers a remarkably subtle discussion of the relationship between geometry and physics while also preempting the failure of familiar geometrical notions at both small and large length scales (hence anticipating aspects of both quantum mechanics and general relativity).

Stein, Howard. "Some Philosophical Pre-History of General Relativity. In *Foundations of Space-Time Theories: Minnesota Studies in the Philosophy of Science*. Edited by John Earman, Clark Glymour, and John Stachel, 3–49. Minneapolis: University of Minnesota Press, 1977.

A magisterial survey of the foundations of physics during the three-hundred-year period between Newton and Einstein, touching on Leibniz, Huygens, Mach, Helmholtz, and Riemann.

Space and Spacetime

Einstein 2002 is a classic (and eminently accessible) historical source for Einstein's own take on the philosophy of physical geometry. Ryckman 2005, Scholz 2013, Brown 2005, and Lehmkuhl 2014 offer commentaries concerning spacetime physics in the 20th century, and Domski 2003 presents a close analysis of Newton's earlier philosophy of geometry. A contemporary philosophical position is advanced by Knox 2013.

Bell, John S. "How to Teach Special Relativity." In *John S. Bell on The Foundations of Quantum Mechanics*. Edited by M. Bell, K. Gottfried, and M. Veltman, 61–73. Singapore: World Scientific Publishing, 2001.

Bell presents a provocative thought experiment to challenge our intuitions regarding the phenomenon of length contraction in special relativity, arguing that classical notions of space and time may still be fit for purpose in modern physics. This is an important argument against the idea that we should attribute characteristic relativistic effects (such as Lorentz contraction and time dilation) to the geometry of Minkowski spacetime. Hence Bell's paper makes a prominent appearance in Brown 2005.

Brown, Harvey R. *Physical Relativity: Space-Time Structure from a Dynamical Perspective*. Oxford: Oxford University Press, 2005.

A detailed and insightful analysis of the philosophical and historical context surrounding the genesis of relativity theory and the purported explanatory significance of coordinate transformations. Coining the term "dynamical relativity", Brown argues that the explanations of relativistic phenomena stem, ultimately, from fundamental dynamical laws, and that the geometry of spacetime is nothing more (and nothing less) than a codification of the *symmetries* of these laws.

Domski, Mary. "The Constructible and the Intelligible in Newton's Philosophy of Geometry." *Philosophy of Science* 70.5 (December 2003): 1114–1124.

Discussing Newton's famous remark in the preface to the Principia that "geometry is founded on mechanical practice," Domski argues that this should not be taken to imply that Newton was entrenched within the prevalent constructivist geometrical tradition of the time, and in particular, that Newton rejected the constructivism of Descartes's *Geometrie*.

Einstein, Albert. "Geometry and Experience." In *The Collected Papers of Albert Einstein*. 208–222. Princeton, N. J: Princeton University Press, 2002.

A landmark lecture delivered to the Prussian Academy of Sciences in 1921, asserting the important role for the distinction between "pure axiomatic geometry" and "practical geometry" in the genesis of relativity. Einstein also discusses geometrical conventionalism, the extension of geometrical notions to small and large length scales, and the possibility of visualizing non-Euclidean geometries.

Knox, Eleanor. "Effective Spacetime Geometry." *Studies in History and Philosophy of Modern Physics* 44.3 (2013): 346–356.

Though the idea that spacetime structure might be emergent from an underlying physics can seem radical, Knox argues that the link between general relativity's geometrical structures and "empirical geometry" indicates that the latter is just the kind of thing that we might regard as emergent (especially with respect to an underlying physics which attributes a role to *torsion*).

Lehmkuhl, Dennis. "Why Einstein Did Not Believe That General Relativity Geometrizes Gravity." *Studies in History and Philosophy of Modern Physics* 46 (2014): 316–326.

In contrast to the received view, Lehmkuhl argues that Einstein did not regard the achievement of general relativity as the "geometrization of gravity" but rather as the *unification* of gravity with inertia. Lehmkuhl argues further that from Einstein's perspective, this unification is not located in the field equations of general relativity but rather in the interpretation of the geodesic equation.

Ryckman, Thomas. *The Reign of Relativity: Philosophy in Physics 1915–1925*. Oxford: Oxford University Press, 2005.

Critiquing the standard narrative inherited from logical positivism, Ryckman contextualizes Einstein's theory against the philosophical background of Kantian and neo-Kantian expressions of transcendental idealism, drawing special attention to the interpretations of general relativity of Hermann Weyl and Arthur S. Eddington. The upshot is a renewed call for a philosophical analysis of the sense that a "geometrized physics" could have.

Scholz, Erhard. "The Problem of Space in the Light of Relativity: The Views of H. Weyl and E. Cartan." In *Eléments d'une biographie de l'espace mathématique*. Edited by L. Bioesmat-Martagon. Nancy, France: Presses Universitaires de Lorraine, 2013.

The "problem of space" is the problem of characterizing physical geometry, or the problem of demarcating candidate descriptions of physical space. This was first tackled in the 19th century by figures such as Riemann, Helmholtz, Lie, and Poincaré. Scholz's paper provides a clear account of how this problem of space arose in a new form after the development of general relativity and how it was addressed in this new context by Weyl and Cartan.

Geometry and Symmetry

Over the history of the concept, symmetry has been taken as: an indication of lawlike regularity, a way of proving the equivalence or equality of magnitudes, a form of invariance, and much more. Symmetry is at the forefront of much contemporary work in physics, and geometrical considerations are paramount in many of these contexts. Yaglom 1988 provides a readable survey of the 19th century analyses of symmetry. Neuenschwander 2017 is a challenging primer on Noether's theorem, one of the fundamental results in this area. Conover 2018 is a brief, easily accessible introduction to Noether's work and career. Brading and Castellani 2013 provides an overview of the use of symmetries in physics through essays from leading scholars.

Brading, Katherine, and Elena Castellani, eds. *Symmetries in Physics: Philosophical Reflections*. Cambridge, UK: Cambridge University Press, 2013.

A definitive recent collection of essays on symmetries in physics, including work on Noether, Eugene Wigner, and many others. Focuses on continuous symmetries, discrete symmetries, symmetry breaking, and problems of interpretation.

Conover, Emily. "Emmy Noether's Vision. *Science News* 193.11 (June 2018): 20.

A brisk, historical, friendly introduction to Noether's career and work.

Neuenschwander, Dwight E. *Emmy Noether's Wonderful Theorem*. Baltimore: Johns Hopkins University Press, 2017.

A challenging, valuable textbook that allows the reader to become familiar with and to work with Noether's theorem. Exercises allow the reader to assess competence with the basic methods, and Neuenschwander provides detailed expositions of all steps in reasoning.

Yaglom, Isaak Moiseevich. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*. Translated by Sergei Sossinsky. Edited by Hardy Grant and Abe Shenitzer. Boston: Birkhäuser, 1988.

Presupposing only high school mathematics, Yaglom's text is a beautifully readable survey of the origins of group theory. The overarching theme is the concept of *symmetry*: its role throughout the history of geometry and its rise to prominence in the development of projective geometry during the 19th century.

Geometrical Formulations of Physical Theories

Almost all major physical theories have been given a "geometric formulation" (often more than once), and the advantages of doing so have been vigorously defended. Lutzen 2005 offers a book-length treatment of Heinrich Hertz's 1894 geometric formulation of classical mechanics, whereas Arnold 2010 (cited under Textbooks) presents a comprehensive geometric treatment of the theory using modern mathematical tools. An analogous treatment of quantum mechanics is presented in Ashtekar and Schilling 1999, and the basis for a geometric approach to electromagnetism is discussed in Belot 1988. Finally, Knox 2011 discusses the significance of geometric formulations of both Newton's and Einstein's theories of gravity.

Ashtekar, Abhay, and Troy A. Schilling. "Geometric Formulation of Quantum Mechanics." In *On Einstein's Path: Essays in Honor of Engelbert Shucking*. Edited by Alex Harvey, 23–66. New York: Springer Science+Business Media, 1999.

Just as symplectic geometry provides a natural geometrical formulation of classical mechanics (see Arnold 2010, cited under Textbooks), Ashtekar and Shilling argue that an analogous geometrical formulation (equipped with a Riemannian metric) is possible for quantum mechanics, leading to important insights that are not accessible in the standard algebraic formulation of the theory.

Belot, Gordon. "Understanding Electromagnetism." ***The British Journal for the Philosophy of Science*** **49.4 (December 1988): 531–555.**

The fact that electromagnetism is a gauge theory (invariant under certain transformations) has led to a nest of interpretive issues connected with the Aharonov-Bohm effect. Belot takes up the question of how best to interpret electromagnetism in particular, and gauge theories more generally, by taking advantage of a natural geometric interpretation of a physical system that tracks the evolution of the system via its path through *phase space*.

Knox, Eleanor. "Newton-Cartan Theory and Teleparallel Gravity: The Force of a Formulation." ***Studies in History and Philosophy of Modern Physics*** **42.4 (2011): 264–275.**

Both general relativity and classical mechanics can be formulated in such a way that gravity is represented traditionally (i.e., non-geometrically), or incorporated into the geometry of the theory. Knox argues that any special feature of gravity in this regard (in either case) cannot be read off from formulations of the theories themselves but depends instead on how the non-gravitational interactions define the appropriate set of inertial frames.

Lutzen, Jesper. ***Mechanistic Images in Geometric Form: Heinrich Hertz's 'Principles of Mechanics.'*** **Oxford: Oxford University Press, 2005.**

Hertz's *Principles of Mechanics* is a highly influential text in 20th-century philosophy of physics, offering an elegant geometrical reformulation of classical mechanics. Lutzen's book-length treatment presents a survey of the mathematical, physical, historical, and philosophical context of Hertz's work, as well as an introduction to the details of *Principles of Mechanics* itself.

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